Multi-asset stochastic volatility

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Global Derivatives – Budapest, May 2016
Outline

- The skew
- The basket skew
- Parametrizing multi-asset stoch vol models
- Tests + mimicking multi-asset local vol
- Correlation swap
- Conclusion

Material of presentation part of Chapter 11 of *Stochastic Volatility Modeling*. See www.lorenzobergomi.com
The ATM(F) skew – see SD IV/book

- Imagine no vanillas exist, have to quote one
  - Delta-hedged option $\Rightarrow$ gamma/theta P&L
  - "Fair " ATM skew given by covariance of spot/future realized variance

- If ATM vanillas traded
  - Consider call spread position centered on ATM such that $\Gamma = 0$
  - Initially $\Gamma = 0$, but $\Gamma + / \Gamma -$ when $S$ moves
  - Trade dynamically ATM vanillas to cancel $\Gamma$
  - Carry P&L: spot/ATM vol cross-gamma + ATM vol gamma. Latter risk smaller

- In liquid markets ATM skew measures implied level of spot/ATM vol covariance; not spot/future realized vol

- Trading intuition expressed by formula at order 1 in vol-of-vol:

$$S_T = \frac{d\hat{\sigma}_{KT}}{d\ln K} \bigg|_S = \frac{1}{2\hat{\sigma}_T^3} \int_0^T \frac{T-t}{T} \langle d\ln S_t \ d\hat{\sigma}_{T,t}^2 \rangle$$

$\hat{\sigma}_{T,t}$: ATM(F)/VS vol at $t$ for maturity $T$

- Realized spot/ATM vol covariance $\Rightarrow$ realized skew
- Market skew $\Rightarrow$ implied spot/ATM vol covariance
  - ATM skew $\Leftrightarrow$ spot/ATM vol covariance: model-independent
  - unlike ATM curvature / vol-of-vol
Short basket skew – 1

Consider short maturity $T \to 0$. Formula simplifies to:

$$S = \frac{1}{2\hat{\sigma}^2} \left\langle d\ln S \, d\hat{\sigma} \right\rangle$$

Consider basket

$$\frac{dB}{B} = \sum_i w_i \frac{dS_i}{S_i} \quad \hat{\sigma}_B^2 = \sum_{ij} w_i w_j \rho_{ij} \hat{\sigma}_i \hat{\sigma}_j$$

Assume homogenous basket: $w_i = \frac{1}{n}$, $\hat{\sigma}_i = \hat{\sigma}$, $\rho_{i\neq j} = \rho_{SS}$

Use formula for ATM skew:

$$\hat{\sigma}_B = \sqrt{\frac{1 + (n-1)\rho_{SS}}{n}} \hat{\sigma}$$

$$S_B = \frac{1 + (n-1)\rho_{SS}}{n} \left( \frac{\hat{\sigma}_B^3}{\hat{\sigma}^3} \right) \left[ \frac{S}{n} + \frac{n-1}{n} \frac{1}{2\hat{\sigma}^2} \left\langle d\ln S \, d\hat{\sigma} \right\rangle_{\text{cross}} \right]$$

Trading interpretation: basket skew generated by spot/vol cross-gamma P&Ls

Parametrize cross spot/vol correl with parameter $\chi_{S\sigma}$:

$$\left\langle d\ln S \, d\hat{\sigma} \right\rangle_{\text{cross}} = \chi_{S\sigma} \left\langle d\ln S \, d\hat{\sigma} \right\rangle$$
Short basket skew – 2

\[ \hat{\sigma}_B = \sqrt{\frac{1 + (n-1)\rho_{SS}}{n}} \hat{\sigma} \]

\[ S_B = \sqrt{\frac{n}{1 + (n-1)\rho_{SS}}} \left[ \frac{1 + (n-1)\chi_{S\sigma}}{n} \right] S \]

⇒ Component’s skew \( S \) only generates \( 1/n \) fraction of basket skew \( S_B \)

⇒ Bulk is generated by cross spot/ATM vol covariance

▶ Large basket: \( n \gg 1 \)

\[ \hat{\sigma}_B \approx \sqrt{\rho_{SS}} \hat{\sigma} \]

\[ S_B \approx \frac{\chi_{S\sigma}}{\sqrt{\rho_{SS}}} S \]

▶ Assume no cross spot/ATM vol correlation: \( \chi_{S\sigma} = 0 \)

\[ S_B \approx 0 \]

▶ Assume cross spot/ATM vol correlation = diagonal spot/ATM vol correlation: \( \chi_{S\sigma} = 1 \)

\[ S_B \approx \frac{1}{\sqrt{\rho_{SS}}} S \geq S \]
Short basket skew – LV model – 1

- In LV model, spot/ATM vol correl = $-1$, so cross spot/ATM vol correl = $-\rho_{SS}$

$$
\langle d \ln S \, d\hat{\sigma} \rangle_{cross} = \rho_{SS} \langle d \ln S \, d\hat{\sigma} \rangle \\
\Rightarrow \chi_{S\sigma} = \rho_{SS}
$$

- For large homogeneous basket, short ATM vol $\hat{\sigma}_B$, skew $S_B$ given by:

$$
\hat{\sigma}_B = \sqrt{\frac{1 + (n - 1)\rho_{SS}}{n}} \hat{\sigma} \approx \sqrt{\rho_{SS}} \hat{\sigma} \\
S_B = \sqrt{\frac{1 + (n - 1)\rho_{SS}}{n}} S \approx \sqrt{\rho_{SS}} S
$$

$\Rightarrow$ Thus

$$
\frac{S_B}{S} \approx \frac{\hat{\sigma}_B}{\hat{\sigma}} \approx \sqrt{\rho_{SS}}
$$

$\Rightarrow$ In LV, basket ATM skew weaker than component’s
Short basket skew – LV model – 2

- Example: \( n = 10, \rho_{SS} = 50\% \)
  - Left: ratios \( \frac{\sigma_B}{\sigma} \) and \( \frac{S_B}{S} \) as a function of maturity (years)
  - Right: ATMF skew expressed as \( \hat{\sigma}_{95\%} - \hat{\sigma}_{105\%} \)

\[
\sqrt{\frac{1+(n-1)\rho_{SS}}{n}} = 74.2\% - \text{dashed line}
\]

\( \Rightarrow \) For short maturities, formula \( \frac{S_B}{S} \approx \sqrt{\rho_{SS}} \) OK
- For longer maturities, basket skew even weaker.

- In reality, \( S_B \geq S \). Typically, \( S_{\text{Index}} \approx 130\% \times S_{\text{Stock}} \)
- Has (mistakenly) convinced some that local/stochastic correlation was needed.
What’s $n$ – how many stocks in a basket?

- Take inhomogeneous basket – what is $n$?

$$
\sigma^2_B = \sigma^2 \sum_{ij} w_i w_j \rho_{ij} = \sigma^2 \left( \sum_i w_i^2 + \rho \sum_{i \neq j} w_i w_j \right) = \sigma^2 \frac{1 + (n^* - 1) \rho}{n^*}
$$

- Effective number of components $n^*$ given by:

$$
n^* = \frac{1}{\sum_i w_i^2}
$$

- With weights as of August 6, 2013:

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P500</th>
<th>Stoxx50</th>
<th>NIKKEI</th>
<th>KOSPI</th>
<th>FTSE</th>
<th>SMI</th>
<th>CAC</th>
<th>Russell2000</th>
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</thead>
<tbody>
<tr>
<td>$n$</td>
<td>500</td>
<td>50</td>
<td>225</td>
<td>200</td>
<td>101</td>
<td>20</td>
<td>40</td>
<td>2000</td>
</tr>
<tr>
<td>$n^*$</td>
<td>143</td>
<td>37</td>
<td>47</td>
<td>17</td>
<td>36</td>
<td>8</td>
<td>20</td>
<td>1056</td>
</tr>
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</table>

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1Also known as the Herfindahl index
Parametrization of multi-asset SV models – 1

- In BS model, gamma P&L only spot/spot $\Rightarrow$ spot/spot correlations all that is needed.

- In SV model, able to account for P&Ls from 3 cross-gammas:
  - $S_i / S_j$
  - $S_i / \hat{\sigma}_j$
  - $\hat{\sigma}_i / \hat{\sigma}_j$

- Parametrization of 2 latter correls should not be model-dependent, or tied to model-specific SV factor structure
  - Physical params that have realized counterparts

$\Rightarrow$ Introduce 2 dimensionless parameters: $\chi_{S\sigma}, \chi_{\sigma\sigma}$:

\[
\rho(S_i, \hat{\sigma}_j^T) = \chi_{S\sigma} \rho(S_i, \hat{\sigma}_i^T)
\]
\[
\rho(\hat{\sigma}_i^T, \hat{\sigma}_j^{T'}) = \chi_{\sigma\sigma} \rho(\hat{\sigma}_i^T, \hat{\sigma}_i^{T'})
\]

- Have assumed that SV params for all components are identical. Could also set:

\[
\rho(S_i, \hat{\sigma}_j^T) = \chi_{S\sigma} \frac{\rho(S_i, \hat{\sigma}_i^T) + \rho(S_j, \hat{\sigma}_j^T)}{2}
\]
\[
\rho(\hat{\sigma}_i^T, \hat{\sigma}_j^{T'}) = \chi_{\sigma\sigma} \frac{\rho(\hat{\sigma}_i^T, \hat{\sigma}_i^{T'}) + \rho(\hat{\sigma}_j^T, \hat{\sigma}_j^{T'})}{2}
\]
Realized values of $\chi_{S\sigma}$, $\chi_{\sigma\sigma}$?

- $\chi_{S\sigma}$ estimated by averaging $\frac{\rho(S_1, \hat{\sigma}_2^T)}{\rho(S_1, \hat{\sigma}_1^T)}$ or $\frac{\rho(S_2, \hat{\sigma}_1^T)}{\rho(S_2, \hat{\sigma}_2^T)}$ over several $T$

- $\chi_{\sigma\sigma}$ estimated by averaging $\frac{\rho(\hat{\sigma}_1^T, \hat{\sigma}_2^{T'})}{\rho(\hat{\sigma}_1^T, \hat{\sigma}_1^{T'})}$ or $\frac{\rho(\hat{\sigma}_2^T, \hat{\sigma}_2^{T'})}{\rho(\hat{\sigma}_2^T, \hat{\sigma}_2^{T'})}$ over several $(T, T')$

Realized values for indices: $T = 3m, 6m, 1y, 2y, ATM$ vols.$^2$

<table>
<thead>
<tr>
<th></th>
<th>June 2008 - June 2013</th>
<th></th>
<th>June 2003 - June 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S&amp;P500</td>
<td>Stoxx50</td>
<td>S&amp;P500</td>
</tr>
<tr>
<td>$\rho$</td>
<td>83%</td>
<td>89%</td>
<td>55%</td>
</tr>
<tr>
<td>$\chi_{S\sigma}$</td>
<td>87%</td>
<td>95%</td>
<td>59%</td>
</tr>
<tr>
<td>$\chi_{S\sigma}$</td>
<td>86%</td>
<td>93%</td>
<td>71%</td>
</tr>
<tr>
<td>$\chi_{\sigma\sigma}$</td>
<td>84%</td>
<td>92%</td>
<td>54%</td>
</tr>
<tr>
<td>$\chi_{\sigma\sigma}$</td>
<td>84%</td>
<td>92%</td>
<td>55%</td>
</tr>
</tbody>
</table>

- Both estimates of $\chi_{\sigma\sigma}$ yield similar values – typical.

  - Vols of vols and vol/vol corrs very stable – stabler than spot vols and spot/spot corrs.

### Parametrization of multi-asset SV models – 3

- **Realized values for (≈ randomly selected) stocks:**

<table>
<thead>
<tr>
<th>June 2008 - June 2013</th>
<th>Bayer</th>
<th>Bayer</th>
<th>Siemens</th>
<th>Sanofi</th>
<th>Eni</th>
<th>Basf</th>
<th>Vinci</th>
<th>Sap</th>
<th>Danone</th>
<th>Apple</th>
<th>JPM</th>
<th>MSFT</th>
<th>Procter</th>
<th>Oracle</th>
<th>Google</th>
<th>GE</th>
<th>GS</th>
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<tbody>
<tr>
<td>ρ</td>
<td>61%</td>
<td>61%</td>
<td>58%</td>
<td>45%</td>
<td>84%</td>
<td>55%</td>
<td>79%</td>
<td>42%</td>
<td></td>
<td>42%</td>
<td>41%</td>
<td>40%</td>
<td>47%</td>
<td>62%</td>
<td>45%</td>
<td>69%</td>
<td>77%</td>
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<tr>
<td>χₜσ</td>
<td>89%</td>
<td>87%</td>
<td>76%</td>
<td>68%</td>
<td>89%</td>
<td>66%</td>
<td>87%</td>
<td>58%</td>
<td></td>
<td>82%</td>
<td>64%</td>
<td>66%</td>
<td>74%</td>
<td>79%</td>
<td>88%</td>
<td>80%</td>
<td>84%</td>
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<tr>
<td>χₜσ</td>
<td>69%</td>
<td>70%</td>
<td>64%</td>
<td>64%</td>
<td>94%</td>
<td>101%</td>
<td>85%</td>
<td>73%</td>
<td></td>
<td>68%</td>
<td>77%</td>
<td>77%</td>
<td>71%</td>
<td>89%</td>
<td>66%</td>
<td>84%</td>
<td>88%</td>
</tr>
<tr>
<td>χₗσ</td>
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<td>74%</td>
<td>69%</td>
<td>67%</td>
<td>85%</td>
<td>51%</td>
<td>82%</td>
<td>69%</td>
<td></td>
<td>69%</td>
<td>71%</td>
<td>78%</td>
<td>77%</td>
<td>83%</td>
<td>76%</td>
<td>83%</td>
<td>83%</td>
</tr>
<tr>
<td>χₗσ</td>
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<td>736%</td>
<td>68%</td>
<td>65%</td>
<td>86%</td>
<td>53%</td>
<td>81%</td>
<td>69%</td>
<td></td>
<td>69%</td>
<td>74%</td>
<td>79%</td>
<td>77%</td>
<td>85%</td>
<td>75%</td>
<td>81%</td>
<td>82%</td>
</tr>
</tbody>
</table>

- **Again narrow range for χₗσₗσ**
- **χₜσ hovers around 70%/80%**

⇒ **Parametrization of multi SV models: spot/spot correlations + χₜσ + χₗσₗσ**

- **Order-1 formula also holds for non-short maturities – assuming basket weights wᵢ are frozen:**

\[
S_B^T = \sqrt{\frac{n}{1 + (n-1)\rho_{SS}}} \left[ 1 + \frac{(n-1)\chi_{t\sigma}}{n} \right] S^T \approx \frac{\chi_{t\sigma}}{\sqrt{\rho_{SS}}} S^T
\]
Numerical tests – mono SV model – 1

▶ Use continuous 2-factor SV model

▶ Dynamics of inst. fwd variances:

\[
\frac{d \xi_T}{\xi_T} = (2\sigma) \alpha_\theta \left( (1 - \theta) e^{-k_1(T-t)} dW^X + \theta e^{-k_2(T-t)} dW^Y \right)
\]

\[
\alpha_\theta = \frac{1}{\sqrt{(1 - \theta)^2 + \theta^2 + 2\rho_{XY} \theta (1 - \theta)}}
\]

▶ Instantaneous vol \( \sigma_T \) of VS vol/ATM vol \( \hat{\sigma}_T \) of maturity \( T \) – for flat TS of VS vols:

\[
\sigma_T = \sigma \alpha_\theta \sqrt{(1 - \theta)^2 I^2(k_1 T) + \theta^2 I^2(k_2 T) + 2\rho_{XY} \theta (1 - \theta) I(k_1 T) I(k_2 T)}
\]

\[
l(x) = \frac{1 - e^{-x}}{x}
\]

▶ ATM skew at order 1 in vol-of-vol for flat term structure of variances:

\[
S_T = \sigma \alpha_\theta \left[ (1 - \theta) \rho_{SX} \frac{k_1 T - (1 - e^{-k_1 T})}{(k_1 T)^2} + \theta \rho_{SY} \frac{k_2 T - (1 - e^{-k_2 T})}{(k_2 T)^2} \right]
\]
Numerical tests – mono SV model – 2

- Params $\theta, \rho_{XY}, k_1, k_2$ have no physical significance. Chosen so that vol of vol $\sigma_T$ matches power-law benchmark

$$\sigma_T^{\text{Benchmark}} = \sigma_0 \left( \frac{T_0}{T} \right)^\alpha$$

$\sigma_T$: vol of VS/ATM vol of maturity $T$

- Different $\rho_{XY}$: equally good fits of benchmark. Example with $\sigma_0 = 100\%$, $T_0 = 3m$, $\alpha = 0.4$:

<table>
<thead>
<tr>
<th>sigma</th>
<th>120.9%</th>
<th>135.8%</th>
<th>174.0%</th>
<th>178.2%</th>
<th>181.9%</th>
<th>185.1%</th>
<th>190.1%</th>
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<tbody>
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<td>theta</td>
<td>57.9%</td>
<td>30.1%</td>
<td>24.5%</td>
<td>23.8%</td>
<td>23.4%</td>
<td>23.1%</td>
<td>22.8%</td>
</tr>
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<td>k1</td>
<td>0.58</td>
<td>2.59</td>
<td>5.35</td>
<td>6.02</td>
<td>6.65</td>
<td>7.26</td>
<td>8.34</td>
</tr>
<tr>
<td>k2</td>
<td>1.19</td>
<td>0.32</td>
<td>0.28</td>
<td>0.27</td>
<td>0.25</td>
<td>0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>rho_XY</td>
<td>-95%</td>
<td>-50%</td>
<td>0%</td>
<td>20%</td>
<td>40%</td>
<td>60%</td>
<td>99%</td>
</tr>
</tbody>
</table>

- No over-parametrization. Different sets $\Rightarrow$ different dynamics for *fwd-starting* vols. Vols of spot-starting vols identical.

- Spot/factor corrs $\rho_{SX}, \rho_{SY}$ such that:
  - Generate given level & term-structure of spot/ATM vol covariances
  - Generate desired skew decay. Typically:

$$S_T \propto \frac{1}{T^\gamma} \quad \text{with} \quad \gamma \in [0.3, 0.7]$$
Numerical tests – multi SV model – 1

▷ Enter $\chi_{S\sigma}$, $\chi_{\sigma\sigma}$

▷ Assume all basket components have same $\theta$, $\rho_{XY}$, $k_1$, $k_2$ params. Set:

\[
\begin{align*}
\rho_{SX}^{\text{cross}} &= \chi_{S\sigma} \rho_{SX} \\
\rho_{SY}^{\text{cross}} &= \chi_{S\sigma} \rho_{SY} \\
\rho_{XX}^{\text{cross}} &= \chi_{\sigma\sigma} \rho_{XX} = \chi_{\sigma\sigma} \\
\rho_{YY}^{\text{cross}} &= \chi_{\sigma\sigma} \rho_{YY} = \chi_{\sigma\sigma} \\
\rho_{XY}^{\text{cross}} &= \chi_{\sigma\sigma} \rho_{XY} = \chi_{\sigma\sigma} \rho_{XY}
\end{align*}
\]

▷ $3n \times 3n$ correl matrix is positive iff $3 \times 3$ matrix:

\[
\begin{pmatrix}
1 & \lambda \rho_{SX} & \lambda \rho_{SY} \\
\lambda \rho_{SX} & 1 & \rho_{XY} \\
\lambda \rho_{SY} & \rho_{XY} & 1
\end{pmatrix}
\]

is positive for (all) 3 values of $\lambda$:

\[
\lambda = \begin{cases} 
1 - \chi_{S\sigma} \\
\frac{1 - \chi_{S\sigma}}{\sqrt{(1 - \rho_{SS})(1 - \chi_{\sigma\sigma})}} \\
\frac{1 + (n - 1) \chi_{S\sigma}}{\sqrt{(1 + (n - 1) \rho_{SS})(1 + (n - 1) \chi_{\sigma\sigma})}}
\end{cases}
\]

▷ $|\chi_{\sigma\sigma}| \leq 1$

▷ $\chi_{S\sigma}$ is allowed to take values larger than 1.
Numerical tests – multi SV model – 2

- Take $n = 10$, $\rho_{SS} = 60\% + \text{params in slide 12, with } \rho_{XY} = 0$.
- Take $\rho_{SX} = -53\%$, $\rho_{SY} = -33.9\%$ for average stock skew ($\approx 70\%$ of index skew).

- Now basket smiles with $\chi_{\sigma\sigma} = 80\%$. 3 values for $\chi_{S\sigma}$: $0\% - 80\% - 115\%$:

\[\Rightarrow \chi_{S\sigma} \text{ does what it’s supposed to do}\]
\[\Rightarrow \text{Contribution of component’s skew to basket skew indeed } \approx \text{ zero.}\]
Numerical tests – multi SV model – 3

- Check formula for basket skew in slide 11. 95/105 skew as a function of $T$ (years):

![Graph showing skew as a function of $T$.]

- Works well
  - With $\rho_{SS} = 60\%$, $\chi_{S\sigma} = 80\%$: $\frac{\chi_{S\sigma}}{\sqrt{\rho_{SS}}} \approx 1$
  - Basket skew $\approx$ stock skew

- Check that $\chi_{\sigma\sigma}$ has no effect on ATM skew – here 1y basket smile, $\chi_{S\sigma} = 80\%$:

![Graph showing ATM skew with varying $\chi_{\sigma\sigma}$.]
Numerical tests – multi SV model – conclusion

- $\chi_{S\sigma}$ indeed drives basket skew
  - With $\rho_{SS} = 60\%$, $\chi_{S\sigma} = 115\%$, ratio of basket skew to component skew is 150%
  - Market index skew $\approx 130\%$ stock skew $\Rightarrow$ implied value of $\chi_{S\sigma} \approx 100\%$.

- $\chi_{\sigma\sigma}$ impacts curvature of basket smile

- Question: what kind of model is multi LV? Can it be mimicked?

- Slide 6: LV model amounts to taking $\chi_{S\sigma} = \rho_{SS}$, $\chi_{\sigma\sigma} = \rho_{SS}$

- Test:
  - Calibrate LV model on component’s smile generate by SV model
  - Generate basket smile in LV with $\rho_{SS} = 60\%$
  - Compare with multi-SV model parametrized with $\chi_{\sigma\sigma} = \chi_{S\sigma} = \rho_{SS}$
Smile indeed similar.

- ATM vols lower in SV model, because of larger vols-of-vols wrt LV.
- In 5y case, pushing $\rho_{SS}$ from 60% to 70% in SV model brings smile up to LV's smile.

⇒ Equivalently, implied spot/spot correl from index smile in SV larger than in LV's
Correlation swap – 1

In (risk-neutral) model, strike of correl swap given by:

\[ \hat{\rho}(T) = E[\rho_{ij}] = \rho_{SS}E \left[ \frac{\int_0^T \sigma^i_t \sigma^j_t dt}{\sqrt{\int_0^T \sigma^2_t dt} \sqrt{\int_0^T \sigma^2_t dt}} \right] = \rho_{SS} \frac{\sigma^i \cdot \sigma^j}{\|\sigma^i\| \|\sigma^j\|} \]

- If inst vols are homothetic \((\sigma^j_t = \lambda \sigma^i_t)\) then \(\hat{\rho}(T) = \rho_{SS}\)
- Otherwise \(\hat{\rho}(T) < \rho_{SS}\)
- \(\sigma^i_t\): object no human has ever seen

Trading interpretation?
- Risk-manage correl swap in BS. Value at \(t\) is:

\[ P = \frac{t \rho_t \sigma_{1t} \sigma_{2t} + (T-t) \hat{\rho}_0 \hat{\sigma}_{1t} \hat{\sigma}_{2t}}{\sqrt{t \sigma^2_{1t} + (T-t) \hat{\sigma}^2_{1t}} \sqrt{t \sigma^2_{2t} + (T-t) \hat{\sigma}^2_{2t}}} \]

- Vega-hegde with var swaps
- Carry P&L is generated by diagonal & cross Vommas.
- Take \(\sigma_{1t} = \hat{\sigma}_{1t}, \sigma_{2t} = \hat{\sigma}_{2t}, \rho_t = \hat{\rho}_0\). P&L of vega-hegded correl swap:

\[ P\&L = \frac{\hat{\rho}_0}{8} \frac{t(T-t)}{T^2} \left( \frac{\delta(\hat{\sigma}^2_{1t})}{\hat{\sigma}^2_{1t}} - \frac{\delta(\hat{\sigma}^2_{2t})}{\hat{\sigma}^2_{2t}} \right)^2 \]

- If relative vol moves identical \(\Rightarrow\) zero P&L
- No sensitivity to realized vols – only to implied vols
Correlation swap – 2

- In multi SV model correl swap is only a function of $\rho_{SS}, \chi_{\sigma\sigma}$ not $\chi_{S\sigma}$.
- At order 2 in vol of vol:

$$\hat{\rho}(T) = \rho_{SS}e^{-(1-\chi_{\sigma\sigma})\sigma^2\frac{1}{T}\int_0^T f(t)dt}$$

with

$$f(t) = \alpha_\theta^2 \left[ (1-\theta)^2 \frac{1-e^{-2k_1 t}}{2k_1 T} \left(1-2 \frac{1-e^{-k_1 (T-t)}}{k_1 T} \right) + \theta^2 \frac{1-e^{-2k_2 t}}{2k_2 T} \left(1-2 \frac{1-e^{-k_2 (T-t)}}{k_2 T} \right) 
+ 2\rho_{XY} \theta (1-\theta) \frac{1-e^{-(k_1+k_2)T}}{(k_1+k_2)T} \left(1-\frac{1-e^{-k_1 (T-t)}}{k_1 T} - \frac{1-e^{-k_2 (T-t)}}{k_2 T} \right) \right]$$

- Example: $n = 10, \rho_{SS} = 60\%$

Correl swap is an exotic vol instrument. If $\rho_{SS}$ known, back out $\chi_{\sigma\sigma}$ from $\hat{\rho}$ – not always possible.

Notice difference with LV
Conclusion

- Can parametrize multi SV model with just 2 additional parameters: $\chi_{S\sigma}, \chi_{\sigma\sigma}$
  - not tied to particular SV model or factor structure.

- Unlike in LV, easy to generate basket skew steeper than component’s skew.

- $\chi_{S\sigma}$ drives basket skew & cross spot/vol correl. $\chi_{S\sigma}^{\text{Realized}} \approx 80\%, \chi_{S\sigma}^{\text{Implied}} \approx 100\%$

- Basket skew $\Rightarrow$ model-indepdt measure of implied cross spot/ATM vol covariance

- $\chi_{\sigma\sigma} \in [0, 1]$. No simple implied counterpart. Realized $\approx 80\%$ pretty stable.

- Life in multi-LV uneventful:
  - LV model has decided that $\chi_{S\sigma} = \chi_{\sigma\sigma} = \rho_{SS}$
  - Only takes spot/spot correls as inputs
  - Implied correlations of equity payoffs (call vs call, correl swap, covar swap) all very close.
    
    Normal:
    $$\hat{\rho} = f(\rho_{SS}) \approx \rho_{SS}$$

- Multi-SV – life in the fast lane:
  - 2 additional parameters control carry levels of cross Vannas & Vommas
  - Implied correlations of equity payoffs (call vs call, correl swap, covar swap) all over the place.
    
    Normal, sensitivities to $\chi_{S\sigma}, \chi_{\sigma\sigma}$ are payoff-dependent:
    $$\hat{\rho} = f(\rho_{SS}, \chi_{S\sigma}, \chi_{\sigma\sigma}) \neq \rho_{SS}$$