Contents

Preface **xv**

1 Introduction **1**
1.1 Characterizing a usable model – the Black-Scholes equation 2
1.2 How (in)effective is delta hedging? 8
1.2.1 The Black-Scholes case 11
1.2.2 The real case 12
1.2.3 Comparing the real case with the Black-Scholes case 14
1.3 On the way to stochastic volatility 15
1.3.1 Example 1: a barrier option 19
1.3.2 Example 2: a forward-start option 22
1.3.3 Conclusion 23
Chapter's digest 24

2 Local volatility **25**
2.1 Introduction – local volatility as a market model 25
2.1.1 SDE of the local volatility model 26
2.2 From prices to local volatilities 27
2.2.1 The Dupire formula 27
2.2.2 No-arbitrage conditions 29
2.2.2.1 Convex order condition for implied volatilities . . 31
2.2.2.2 Implied volatilities of general convex payoffs . . . 32
2.3 From implied volatilities to local volatilities 33
2.3.1 Dividends 34
2.3.1.1 An exact solution 35
2.3.1.2 An approximate solution 36
2.4 From local volatilities to implied volatilities 38
2.4.1 Implied volatilities as weighted averages of instantaneous volatilities 39
2.4.2 Approximate expression for weakly local volatilities 41
2.4.3 Expanding around a constant volatility 44
2.4.4 Discussion 44
2.4.5 The smile near the forward 46
2.4.5.1 A constant local volatility function 47
2.4.5.2 A power-law-decaying ATMF skew 47
2.4.6 An exact result for short maturities 47

www.lorenzobergomi.com

2.5 The dynamics of the local volatility model . . . 49
2.5.1 The dynamics for strikes near the forward . . . 49
2.5.2 The Skew Stickiness Ratio (SSR) . . . 51
2.5.2.1 The $\mathcal{R} = 2$ rule . . . 53
2.5.3 The $\mathcal{R} = 2$ rule is exact . . . 53
2.5.3.1 Time-independent local volatility functions . . . 53
2.5.3.2 Short maturities . . . 56
2.5.4 SSR for a power-law-decaying ATMF skew . . . 56
2.5.5 Volatilities ofvolatilities . . . 57
2.5.6 Examples and discussion . . . 58
2.5.7 SSR in local and stochastic volatility models . . . 61
2.6 Future skews and volatilities of volatilities . . . 63
2.6.1 Comparison with stochastic volatility models . . . 65
2.7 Delta and carry P&L . . . 66
2.7.1 The "local volatility delta" . . . 66
2.7.2 Using implied volatilities – the sticky-strike delta Δ^{SS} . . . 67
2.7.3 Using option prices – the market-model delta Δ^{MM} . . . 70
2.7.4 Consistency of Δ^{SS} and Δ^{MM} . . . 71
2.7.5 Local volatility as the simplest market model . . . 72
2.7.6 A metaphor of the local volatility model . . . 74
2.7.7 Conclusion . . . 75
2.7.8 Appendix – delta-hedging only . . . 77
2.7.9 Appendix – the drift of V_t in the local volatility model . . . 78
2.8 Digression – using payoff-dependent break-even levels . . . 79
2.9 The vega hedge . . . 81
2.9.1 The vanilla hedge portfolio . . . 81
2.9.2 Calibration and its meaningfulness . . . 85
2.10 Markov-functional models . . . 85
2.10.1 Relationship of Gaussian copula to multi-asset local volatility prices . . . 87
Appendix A – the Uncertain Volatility Model . . . 87
A.1 An example . . . 89
A.2 Marking to market . . . 90
A.2.1 An unhedged position . . . 90
A.2.2 A hedged position – the λ-UVM . . . 90
A.2.3 Discussion . . . 92
A.3 Using the UVM to price transaction costs . . . 93
Chapter's digest . . . 96

3 Forward-start options **103**
3.1 Pricing and hedging forward-start options . . . 103
3.1.1 A Black-Scholes setting . . . 104
3.1.2 A vanilla portfolio whose vega is independent of S . . . 105
3.1.3 Digression: replication of European payoffs . . . 106
3.1.4 A vanilla hedge . . . 107

www.lorenzobergomi.com

3.1.5 Using the hedge in practice – additional P&Ls . . . 108
3.1.5.1 Before T_1– volatility-of-volatility risk . . . 108
3.1.5.2 At T_1 – forward-smile risk . . . 110
3.1.6 Cliquet risks and their pricing: conclusion . . . 111
3.1.7 Lower/upper bounds on cliquet prices from the vanilla smile 113
3.1.8 Calibration on the vanilla smile – conclusion . . . 116
3.1.9 Forward volatility agreements . . . 116
3.1.9.1 A vanilla portfolio whose vega is linear in S . . . 117
3.1.9.2 Additional P&Ls and conclusion . . . 118
3.2 Forward-start options in the local volatility model . . . 119
3.2.1 Approximation for $\widehat{\sigma}_T$ . . . 119
3.2.2 Future skews in the local volatility model . . . 124
3.2.3 Conclusion . . . 125
3.2.4 Vega hedge of a forward-start call in the local volatility model 125
3.2.5 Discussion and conclusion . . . 128
Chapter's digest . . . 130

4 Stochastic volatility – introduction 133
4.1 Modeling vanilla option prices . . . 133
4.1.1 Modeling implied volatilities . . . 134
4.2 Modeling the dynamics of the local volatility function . . . 135
4.2.1 Conclusion . . . 139
4.3 Modeling implied volatilities of power payoffs . . . 140
4.3.1 Implied volatilities of power payoffs . . . 140
4.3.2 Forward variances of power payoffs . . . 143
4.3.3 The dynamics of forward variances . . . 144
4.3.4 Markov representation of the variance curve . . . 145
4.3.5 Dynamics for multiple variance curves . . . 147
4.3.6 The log contract, again . . . 148
Chapter's digest . . . 150

5 Variance swaps 151
5.1 Variance swap forward variances . . . 151
5.2 Relationship of variance swaps to log contracts . . . 153
5.2.1 A simple formula for $\widehat{\sigma}_{\mathrm{VS},T}$ . . . 155
5.3 Impact of large returns . . . 156
5.3.1 In diffusive models . . . 156
5.3.2 In jump-diffusion models . . . 158
5.3.3 Difference of VS and log-contract implied volatilities . . . 160
5.3.4 Impact of the skewness of daily returns – model-free . . . 162
5.3.5 Inferring the skewness of daily returns from market smiles? 163
5.3.6 Preliminary conclusion . . . 164
5.3.7 In reality . . . 164
5.4 Impact of strike discreteness . . . 167
5.5 Conclusion . . . 168

www.lorenzobergomi.com

5.6 Dividends . . . 171
5.6.1 Impact on the VS payoff . . . 171
5.6.2 Impact on the VS replication . . . 172
5.7 Pricing variance swaps with a PDE . . . 173
5.8 Interest-rate volatility . . . 175
5.9 Weighted variance swaps . . . 176
Appendix A – timer options . . . 180
A.1 Vega/gamma relationship in the Black-Scholes model . . . 181
A.2 Model-independent payoffs based on quadratic variation . . . 183
A.3 How model-independent are timer options? . . . 185
A.4 Leveraged ETFs . . . 187
Appendix B – perturbation of the lognormal distribution . . . 188
B.1 Perturbing the cumulant-generating function . . . 190
B.2 Choosing a normalization and generating a density . . . 191
B.3 Impact on vanilla option prices and implied volatilities . . . 192
B.4 The ATMF skew . . . 193
Chapter's digest . . . 195

6 An example of one-factor dynamics: the Heston model 201
6.1 The Heston model . . . 201
6.2 Forward variances in the Heston model . . . 202
6.3 Drift of V_t in first-generation stochastic volatility models . . . 203
6.4 Term structure of volatilities of volatilities in the Heston model . . . 204
6.5 Smile of volatility of volatility . . . 205
6.6 ATMF skew in the Heston model . . . 206
6.6.1 The smile at order one in volatility of volatility . . . 207
6.6.2 Example . . . 211
6.6.3 Term structure of the ATMF skew . . . 212
6.6.4 Relationship between ATMF volatility and skew . . . 213
6.7 Discussion . . . 213
Chapter's digest . . . 216

7 Forward variance models 217
7.1 Pricing equation . . . 217
7.2 A Markov representation . . . 219
7.3 N-factor models . . . 221
7.3.1 Simulating the N-factor model . . . 222
7.3.2 Volatilities and correlations of variances . . . 223
7.3.3 Vega-hedging in finite-dimensional models . . . 225
7.4 A two-factor model . . . 226
7.4.1 Term structure of volatilities of volatilities . . . 228
7.4.2 Volatilities and correlations of forward variances . . . 229
7.4.3 Smile of VS volatilities . . . 231
7.4.4 Non-constant term structure of VS volatilities . . . 232
7.4.5 Conclusion . . . 233

www.lorenzobergomi.com

7.5 Calibration – the vanilla smile . . . 234
7.6 Options on realized variance . . . 235
7.6.1 A simple model (SM) . . . 236
7.6.2 Preliminary conclusion . . . 239
7.6.3 Examples . . . 239
7.6.4 Accounting for the term structure of VS volatilities . . . 241
7.6.5 Vega and gamma hedges . . . 242
7.6.6 Examples . . . 244
7.6.7 Non-flat VS volatilities . . . 249
7.6.8 Accounting for the discrete nature of returns . . . 249
7.6.9 Conclusion . . . 253
7.6.10 What about the vanilla smile? Lower and upper bounds . . . 254
7.6.11 Options on forward realized variance . . . 260
7.7 VIX futures and options . . . 261
7.7.1 Modeling VIX smiles in the two-factor model . . . 263
7.7.2 Simulating VIX futures in the two-factor model . . . 268
7.7.3 Options on VIX ETFs/ETNs . . . 269
7.7.4 Consistency of S&P 500 and VIX smiles . . . 273
7.7.5 Correlation structure of VIX futures . . . 276
7.7.6 Impact on smiles of options on realized variance . . . 277
7.7.7 Impact on the vanilla smile . . . 278
7.8 Discrete forward variance models . . . 278
7.8.1 Modeling discrete forward variances . . . 279
7.8.2 Direct modeling of VIX futures . . . 283
7.8.3 A dynamics for S_t . . . 287
7.8.4 The vanilla smile . . . 291
7.8.5 Conclusion . . . 297
Chapter's digest . . . 300

8 The smile of stochastic volatility models 307
8.1 Introduction . . . 307
8.2 Expansion of the price in volatility of volatility . . . 308
8.3 Expansion of implied volatilities . . . 313
8.4 A representation of European option prices in diffusive models . . . 316
8.4.1 Expansion at order one in volatility of volatility . . . 318
8.4.2 Materializing the spot/volatility cross-gamma P&L . . . 320
8.5 Short maturities . . . 321
8.5.1 Lognormal ATM volatility – SABR model . . . 323
8.5.2 Normal ATM volatility – Heston model . . . 324
8.5.3 Vanishing correlation – a measure of volatility of volatility 325
8.6 A family of one-factor models – application to the Heston model . 325
8.7 The two-factor model . . . 326
8.7.1 Uncorrelated case . . . 328
8.7.2 Correlated case – the ATMF skew and its term structure . . . 328
8.8 Conclusion . . . 333

www.lorenzobergomi.com

8.9 Forward-start options – future smiles . . . 333
8.10 Impact of the smile of volatility of volatility on the vanilla smile . . 335
Appendix A – Monte Carlo algorithms for vanilla smiles . . . 336
A.1 The mixing solution . . . 336
A.2 Gamma/theta P&L accrual . . . 338
A.3 Timer option-like algorithm . . . 340
A.4 A comparison . . . 342
A.5 Dividends . . . 344
A.5.1 An efficient approximation . . . 345
Appendix B – local volatility function of stochastic volatility models . . 345
Appendix C – partial resummation of higher orders . . . 347
Chapter's digest . . . 351

9 Linking static and dynamic properties of stochastic volatility models 357
9.1 The ATMF skew . . . 357
9.2 The Skew Stickiness Ratio (SSR) . . . 358
9.3 Short-maturity limit of the ATMF skew and the SSR . . . 359
9.4 Model-independent range of the SSR . . . 359
9.5 Scaling of ATMF skew and SSR – a classification of models . . . 361
9.6 Type I models – the Heston model . . . 362
9.7 Type II models . . . 363
9.8 Numerical evaluation of the SSR . . . 368
9.9 The SSR for short maturities . . . 368
9.10 Arbitraging the realized short SSR . . . 370
9.10.1 Risk-managing with the lognormal model . . . 370
9.10.2 The realized skew . . . 372
9.10.3 Splitting the theta into three pieces . . . 372
9.10.4 Backtesting on the Euro Stoxx 50 index . . . 373
9.10.5 The "fair" ATMF skew . . . 376
9.10.6 Relevance of model-independent properties . . . 378
9.11 Conclusion . . . 378
9.11.1 SSR in local and stochastic volatility models – and in reality 379
9.11.2 Volatilities of volatilities . . . 382
9.11.3 Carry P&L of a partially vega-hedged position . . . 383
Chapter's digest . . . 386

10 What causes equity smiles? 391
10.1 The distribution of equity returns . . . 391
10.1.1 The conditional distribution . . . 394
10.2 Impact of the distribution of daily returns on derivative prices . . . 395
10.2.1 A stochastic volatility model with fat-tailed returns . . . 396
10.2.2 Vanilla smiles . . . 399
10.2.3 Discussion . . . 401
10.2.4 Variance swaps . . . 404

www.lorenzobergomi.com

10.2.5 Daily cliquets . . . 405
10.3 Conclusion . . . 406
Appendix A – jump-diffusion/Lévy models . . . 407
A.1 A stress-test reserve/remuneration policy . . . 407
A.2 Pricing equation . . . 409
A.3 ATMF skew . . . 411
A.4 Jump scenarios in calibrated models . . . 414
A.5 Lévy processes . . . 416
A.6 Conclusion . . . 416
Chapter's digest . . . 418

11 Multi-asset stochastic volatility **421**
11.1 The short ATMF basket skew . . . 421
11.1.1 The case of a large homogeneous basket . . . 422
11.1.2 The local volatility model . . . 423
11.1.3 The basket skew in reality . . . 424
11.1.4 Digression – how many stocks are there in an index? . . . 425
11.2 Parametrizing multi-asset stochastic volatility models . . . 425
11.2.1 A homogeneous basket . . . 426
11.2.2 Realized values of $\chi_{S\sigma}$ and $\chi_{\sigma\sigma}$ . . . 430
11.3 The ATMF basket skew . . . 431
11.3.1 Application to the two-factor model . . . 434
11.3.2 Numerical examples . . . 435
11.3.3 Mimicking the local volatility model . . . 438
11.4 The correlation swap . . . 440
11.4.1 Approximate formula in the two-factor model . . . 442
11.4.2 Examples . . . 444
11.5 Conclusion . . . 445
Appendix A – bias/standard deviation of the correlation estimator . . . 446
Chapter's digest . . . 449

12 Local-stochastic volatility models **453**
12.1 Introduction . . . 453
12.2 Pricing equation and calibration . . . 454
12.2.1 Pricing . . . 454
12.2.2 Is it a price? . . . 455
12.2.3 Calibration to the vanilla smile . . . 456
12.2.4 PDE method . . . 457
12.2.5 Particle method . . . 462
12.3 Usable models . . . 463
12.3.1 Carry P&L . . . 463
12.3.2 P&L of a hedged position . . . 465
12.3.3 Characterizing usable models . . . 468
12.4 Dynamics of implied volatilities . . . 470
12.4.1 Components of the ATMF skew . . . 471

www.lorenzobergomi.com

12.4.2 Dynamics of ATMF volatilities 474
12.4.2.1 SSR 475
12.4.2.2 Volatilities of volatilities 477
12.4.3 Numerical evaluation of the SSR and volatilities of volatilities 477
12.5 Numerical examples 478
12.6 Discussion 482
12.6.1 Future smiles in mixed models 485
12.7 Conclusion 488
Appendix A – alternative schemes for the PDE method 489
Chapter's digest 493

Epilogue **495**

Bibliography **497**

Index **503**

www.lorenzobergomi.com