Local/stochastic volatility models – and non-models

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Intro

- LV model - see 1st talk – is a market model for spot/vanilla options.
  - Takes as input any non-arb vanilla smile
  - Joint dynamics of spot/implied vols set by smile used for calibration.

- Stoch vol models afford a handle on dynamics of implied volatilities
  - But, market models only for one-dimensional set of volatility instruments: term structure of VS or ATMF volatilities.

- Poor man’s fix: mix both so that, hopefully:
  - is a market model for spot/vanilla options
  - affords some leverage on joint dynamics of spot/implied vols – through SV parameters.

- Typically decorate SV model by multiplying SV instantaneous vol by local vol component.

- Is it a model?

- Provided answer is positive:
  - What is the delta? What are the vegas?
  - What kind of model is it?

(Material of this presentation part of forthcoming book)
Practically usable models

- Define assets $A_i$ to be used as hedging instruments – pricing function $P(t, A_1, \ldots, A_n)$

- First order contribution to P&L cancelled by delta hedging:

$$P&L = -[P(t + \delta t, A + \delta A) - P(t, A)] + \sum \frac{dP}{dA_i} (\delta A_i - \mu_i A_i \delta t)$$

- First non-trivial contribution to P&L – order 1 in $\delta t$, 2 in $\delta A$:

$$P&L = -\frac{dP}{dt} - \frac{1}{2} \sum \frac{d^2P}{dA_i dA_j} \delta A_i \delta A_j$$

- Condition for non-nonsensical P&L: there exists a positive breakeven covariance matrix $C_{ij}(t, A)$ such that:

$$\frac{dP}{dt} = \frac{1}{2} \sum \frac{d^2P}{dA_i dA_j} A_i A_j C_{ij}$$

$$P&L = -\frac{1}{2} \sum A_i A_j \frac{d^2P}{dA_i dA_j} \left( \frac{\delta A_i}{A_i} \frac{\delta A_j}{A_j} - C_{ij} \delta t \right)$$

- Trademark of a market model
- Sign of P&L depends on mismatch between model & realized covariances

- Ideally we would like to set the $C_{ij}$ – otherwise may need to use implied levels.
Models used as examples in presentation

- **Mixed Heston model**

  \[
  \begin{align*}
  dS_t &= (r - q)S_t dt + \sigma(t, S_t)\sqrt{V_t} S_t dW_t \\
  dV_t &= -k(V_t - V^0)dt + \nu \sqrt{V_t} dZ_t
  \end{align*}
  \]

- **Mixed two-factor model**

  \[
  \begin{align*}
  dS_t &= (r - q)S_t dt + \sigma(t, S_t)\sqrt{\zeta_t} S_t dW^S_t \\
  \frac{d\zeta_t}{\zeta_t} &= 2\nu \alpha \theta \left( (1 - \theta) e^{-k_1(T-t)} dW^1_t + \theta e^{-k_2(T-t)} dW^2_t \right)
  \end{align*}
  \]

  where \( \alpha \theta = 1/\sqrt{(1 - \theta)^2 + \theta^2 + 2\rho \theta (1 - \theta)}; \nu \) vol of short vol.

- **LV component** \( \sigma(t, S) \) calibrated on vanilla smile.

- **Pricing function in mixed model:**
  - in Heston model \( P^M(t, S, \sigma, V) \), \( V \) number.
  - in two-factor model \( P^M(t, S, \sigma, \zeta^u) \), \( \zeta^u \) curve.
Usage of mixed models

- Choose model parameters & initial values of state variables:
  - In Heston: \((k, \sigma, \rho, V^0), \, V\).
  - In two-factor model: \((k_1, k_2, \theta, \nu, \rho_{12}, \rho_{S1}, \rho_{S2}), \, \zeta\).

- Calibrate local volatility function \(\sigma(t, S)\) to market smile.
  - In 1-factor model like Heston: solve fwd PDE for density.

- Then Shift+F9 \(\Rightarrow\) produces a (real) number. Is it a price?

- What about deltas?
  - Typically, move spot, recalibrate local vol and reprice.
  - Is it right delta? What kind of carry P&L does this materialize?

- Let’s assume this is a model. Can we have an approximate way of sizing up:
  - volatilities of implied vols
  - covariances of spot and implied vols – equivalently SSR?
Two pricing functionals

- $P^M(t,x)$: takes as inputs $t, S, LV$ function + state variables $\lambda$ of SV model:
  
  $$P^M(t, S, \sigma(,), \lambda)$$

  - In Heston: $\lambda = V$ – number
  - In two-factor model: $\lambda = \zeta^u$ – curve

- $P(t, \hat{x})$ takes as inputs $t, S, implied$ $vols$ + state variables of SV model:
  
  $$P(t, S, \hat{\sigma}_{KT}, \lambda)$$

- Could include in $x, \hat{x}$ model parameters ($\equiv$ state variables with zero drift/vol).

  - Could use prices rather than implied vols.
Carry P&L – $P(t, S, \hat{\sigma}_{KT}, \lambda)$

- In mixed model – for a set LV function – $\hat{\sigma}_{KT}$ is a function of $t, S, \sigma(,) +$ state variables: $\hat{x} = \hat{x}(t, x)$:

  $$P^M(t, x) = P(t, \hat{x}(t, x))$$

- Implied vols given by:

  $$\hat{\sigma}_{KT} \equiv \Sigma^M_{KT}(t, S, \sigma, \lambda)$$

  $P^M, P$ related through:

  $$P^M(t, S, \sigma, \lambda) = P(t, S, \Sigma^M_{KT}(t, S, \sigma, \lambda), \lambda)$$

- Pricing equation for $P^M$ – with set LV function – zero rates:

  $$\frac{dP^M}{dt} + \left( \Sigma_k \mu_k \frac{d}{dx_k} + \frac{1}{2} \Sigma_{kl} a_{kl} \frac{d^2}{dx_k dx_l} \right) P^M = 0$$
Carry P&L – 2

- Switch to variables \( \hat{x} \):
  \[
  \frac{dP}{dt} + \left( \sum_i \hat{\mu}_i \frac{d}{d\hat{x}_i} + \frac{1}{2} \sum_{ij} \hat{a}_{ij} \frac{d^2}{d\hat{x}_i d\hat{x}_j} \right) P = 0
  \]
  with:
  \[
  \begin{align*}
  \hat{\mu}_i &= \frac{d\hat{x}_i}{dt} + \sum_{k} \mu_k \frac{d\hat{x}_i}{dx_k} + \frac{1}{2} \sum_{kl} a_{kl} \frac{d^2\hat{x}_i}{dx_k dx_l} \\
  \hat{a}_{ij} &= \sum_{kl} a_{kl} \frac{d\hat{x}_i}{dx_k} \frac{d\hat{x}_j}{dx_l}
  \end{align*}
  \]

- \( \hat{\mu}_i \) drift of \( \hat{x}_i \) and \( \hat{a}_{ij} \) covariance matrix of \( \hat{x}_i \) and \( \hat{x}_j \)
  - as generated by mixed model with fixed LV function.
  - \( \frac{d\hat{x}_i}{dx_k} \) involve derivatives of functional \( \Sigma^M_{KT}(t, S, \sigma, \lambda) \) with respect to \( t, S, \lambda \).

- Now consider P&L of short option position – unhedged for now – zero rates:
  \[
  P\&L = - P(t + \delta t, \hat{x} + \delta\hat{x}) + P(t, \hat{x})
  \]
Expand at order two in $\delta \hat{x}$, one in $\delta t$.

\[
P&L = - \frac{dP}{dt} \delta t - \sum_i \frac{dP}{d\hat{x}_i} \delta \hat{x}_i - \frac{1}{2} \sum_{ij} \frac{d^2P}{d\hat{x}_i d\hat{x}_j} \delta \hat{x}_i \delta \hat{x}_j
\]

\[
= - \sum_i \frac{dP}{d\hat{x}_i} (\delta \hat{x}_i - \hat{\mu}_i \delta t) - \frac{1}{2} \sum_{ij} \frac{d^2P}{d\hat{x}_i d\hat{x}_j} (\delta \hat{x}_i \delta \hat{x}_j - \hat{a}_{ij} \delta t)
\]

Among components of $\hat{x}$:
- $O_i$ market observables: $S, \hat{\sigma}_{KT}$.
- $\lambda_k$ = state variables of SV model

Rewrite P&L:

\[
P&L = - \sum_i \frac{dP}{dO_i} (\delta O_i - \hat{\mu}_i \delta t) - \frac{1}{2} \sum_{ij} \frac{d^2P}{dO_i dO_j} (\delta O_i \delta O_j - \hat{a}_{ij} \delta t)
\]

\[
- \sum_k \frac{dP}{d\lambda_k} (\delta \lambda_k - \hat{\mu}_k \delta t)
\]

\[
- \frac{1}{2} \sum_{kl} \frac{d^2P}{d\lambda_k d\lambda_l} (\delta \lambda_k \delta \lambda_l - \hat{\omega}_{kl} \delta t) - \sum_{ik} \frac{d^2P}{dO_i d\lambda_k} (\delta O_i \delta \lambda_k - \hat{a}_{ik} \delta t)
\]
P&L hedged portfolio

- Portfolio: option + hedges that offset sensitivities $\frac{dP}{dO_i}$: $P_H = P + \sum_i \alpha_i f_i(t, S, O_i)$
- P&L equation also holds for hedge instruments $\Rightarrow$ canceling $\delta O_i$ term cancels $\hat{\mu}_i \delta t$ contribution. P&L of hedged position:

$$P&L_H = -\frac{1}{2} \sum_{ij} \frac{d^2 P_H}{dO_i dO_j} (\delta O_i \delta O_j - \hat{a}_{ij} \delta t) - \sum_k \frac{dP_H}{d\lambda_k} (\delta \lambda_k - \hat{\mu}_k \delta t) - \frac{1}{2} \sum_{kl} \frac{d^2 P_H}{d\lambda_k d\lambda_l} (\delta \lambda_k \delta \lambda_l - \hat{a}_{kl} \delta t) - \sum_{ik} \frac{d^2 P_H}{dO_i d\lambda_k} (\delta O_i \delta \lambda_k - \hat{a}_{ik} \delta t)$$

- 1st piece OK: thetas matching gammas on market instruments.
  - $\hat{a}_{ij}$ positive covariance matrix: $\hat{a}_{ij} = \sum_{kl} a_{kl} \frac{d\tilde{x}_i}{d\lambda_k} \frac{d\tilde{x}_j}{d\lambda_l}$
- 2nd / 3d pieces no good. P&L leakage from variation of SV state variables.

- By construction value of hedges indpdt on $\lambda_k$: $\frac{df_i}{d\lambda_k} = 0$, so:

$$\frac{dP_H}{d\lambda_k} = \frac{dP}{d\lambda_k}, \quad \frac{d^2 P_H}{d\lambda_k d\lambda_l} = \frac{d^2 P}{d\lambda_k d\lambda_l}, \quad \frac{d^2 P_H}{dO_i d\lambda_k} = \frac{d^2 P}{dO_i d\lambda_k}$$
P&L hedged portfolio – 2

- $\delta\lambda_k$ are not market values – are in our control. For example, take $\delta\lambda_k = \hat{\mu}_k \delta t$.

- Still leaves us with 3d piece in P&L:

$$P&L_{\text{H}}^{\text{leak}} = -\frac{1}{2} \sum_{kl} \frac{d^2 P}{d\lambda_k d\lambda_l} (\delta\lambda_k \delta\lambda_l - \hat{a}_{kl} \delta t) - \sum_{ik} \frac{d^2 P}{dO_i d\lambda_k} (\delta O_i \delta\lambda_k - \hat{a}_{ik} \delta t)$$

- Is there a solution to P&L leakage?

- YES – need condition on $P(t, O, \lambda)$:

$$\left. \frac{dP}{d\lambda_k} \right|_{S, \hat{\sigma}_{KT}} = 0, \forall k$$

- Pricing functional $P(t, S, \hat{\sigma}_{KT}, \lambda)$ must have zero sensitivity to SV state variables.
Conclusion: admissible (or gauge-invariant) models

- Criterion for models that can be used in trading: \( P(t, S, \hat{\sigma}_{KT}, \lambda) \):

\[
\frac{dP}{d\lambda_k} \bigg|_{S, \hat{\sigma}_{KT}} = 0
\]

- \( P&L_H \) of delta-hedged/vega-hedged position then has typical form of market models:

\[
P&L_H = -\frac{1}{2} \sum_{ij} \frac{d^2P}{dO_idO_j} \left( \delta O_i \delta O_j - \hat{a}_{ij} \delta t \right)
\]

Break-even covariance levels are given by covariances in model with fixed LV function: \( \hat{a}_{ij} = \sum_{kl} a_{kl} \frac{dx_i}{dx_k} \frac{dx_j}{dx_l} \).

- Pbm: condition \( \frac{dP}{d\lambda} \bigg|_{S, \hat{\sigma}_{KT}} = 0 \) usually not satisfied.
  - Ex: not satisfied in local/stoch vol model built on Heston model:
    \[
    \frac{d}{dV} P(t, S, \hat{\sigma}_{KT}, V) \neq 0 \quad \Rightarrow \quad P&L \text{ leakage}
    \]
  - Not usable in trading.

- Do admissible models exist at all?
  - YES.
Admissible models – 2

- Consider mixed two-factor model. Pricing function $P(t, S, \hat{\sigma}_{KT}, \zeta^u)$.

- Model equivalently written as:

\[
\begin{align*}
    dS_t &= (r - q)S_t dt + \sqrt{\zeta^u} \sqrt{f(t, X^1_t, X^2_t)} \sigma(t, S_t) S_t dW^S_t \\
    dX^1_t &= -k_1 X^1_t dt + dW^1_t \\
    dX^2_t &= -k_2 X^2_t dt + dW^2_t
\end{align*}
\]

with $X^1_0 = 0, X^2_0 = 0$ and:

\[
f(t, x_1, x_2) = e^{2\nu_x \theta (1-\theta) x_1 + \theta x_2} - \frac{(2\nu_x \theta)^2}{2} \chi(t)
\]

\[
\chi(t) = (1 - \theta)^2 \frac{1 - e^{-2k_1 t}}{2k_1} + \theta^2 \frac{1 - e^{-2k_2 t}}{2k_2} + 2\rho \theta (1 - \theta) \frac{1 - e^{-(k_1 + k_2) t}}{k_1 + k_2}
\]

- Pick arbitrary $\varphi^u$, do following transformation:

\[
\begin{align*}
    \zeta^u &\rightarrow \varphi^u \zeta^u \\
    \sigma(u, S) &\rightarrow \sqrt{\frac{1}{\varphi^u}} \sigma(u, S)
\end{align*}
\]

- SDEs for $S_t, X^1_t, X^2_t$ unchanged: $\frac{\delta P}{\delta \zeta^u} = 0 \Rightarrow$ mixed two-factor model admissible.
Admissible models – 3

- Other admissible models:
  - lognormal model for $V_t$ (SABR)
  - smiled version of two-factor model (see SD III)

- Significance of condition $\frac{dP}{d\lambda} \bigg|_{S,\sigma_{KT}} = 0$
  - $\frac{dP}{d\lambda} \bigg|_{S,\sigma_{KT}} \neq 0$: price depends on more state variables than hedge instruments. Ex. with Heston model: $P(t, S, \sigma_{KT}, V)$.

- State variables $\lambda$ are stochastic $\Rightarrow$ model allocates thetas proportional to $\frac{d^2P}{d\lambda^2}$, $\frac{d^2P}{d\lambda d\Omega}$ $\Rightarrow$ P&L leakage, even if $\delta\lambda = 0$.

- Does not happen with model parameters $V^0$, $k$, $\nu$, $\rho$ – do not generate P&L leakage.
  - Model allocates no theta to gammas on model params.
  - Like making $P$ a function of a non-financial state variable – e.g. temperature.

- In admissible models, SV degrees of freedom do impact dynamics of assets, yet do not require extra hedges.
Now know which models are usable – what’s left to do?

- Size up break-even covariance levels for $S/\hat{\sigma}_{KT}$, $\hat{\sigma}_{KT}/\hat{\sigma}_{K'T'}$.
  - Like them, use model; don’t like them, don’t use model.
  - In practice, look at dynamics of implied vols with floating strikes – fixed moneyness, rather than fixed strikes.

- Approximate formulae for vols of vols and spot/vol covariances – for ATMF vols?

- Will consider in particular SSR:

$$\mathcal{R}_T = \frac{1}{S_T} \frac{\langle d\hat{\sigma}_T d\ln S \rangle}{\langle (d\ln S)^2 \rangle}$$

SSR $\Leftrightarrow$ Spot/ATMF vol covariance
Perturbation in mixed models

- Use mixed 2F model. Take $\zeta^u = 1 \forall u$ – OK since admissible model: $\frac{\delta P}{\delta \zeta^u} = 0$.

$$dS_t = (r - q)S_t \, dt + \sigma(t, S_t) \sqrt{f(t, X^1_t, X^2_t)} \, S_t \, dW_t^S$$

- Expand at order 1 in (a) LV function, (b) vol of vol $\nu$.

  - LV: Take

$$\sigma(t, S) = \sigma_0 + \delta \sigma(t, S)$$

with

$$\delta \sigma(t, S) = \alpha(t) x, \quad x = \ln \left( \frac{S}{F_t} \right)$$

  - SV: Expand $f$ at order 1 in $\nu$:

$$\sqrt{f(t, X^1_t, X^2_t)} = 1 + \frac{\nu}{2} g(t, X^1_t, X^2_t)$$

$$g(t, x_1, x_2) = 2\alpha \theta [(1 - \theta) x_1 + \theta x_2]$$

- Write solution of SDE:

$$dS_t = (r - q)S_t \, dt + \left( \sigma_0 + \delta \sigma(t, S_t) \right) \left( 1 + \frac{\nu}{2} g(t, X^1_t, X^2_t) \right) \, S_t \, dW_t^S$$

as:

$$S_t = S^0_t + \delta S_t^{LV} + \delta S_t^{SV}$$
Perturbation in mixed models – 2

\[
\begin{align*}
    dS_t^0 &= (r - q)S_t^0 dt + \sigma_0 S_t^0 dW_t^S & S_{t=0}^0 &= S_0 \\
    d\delta S_t^{LV} &= (r - q)\delta S_t^{LV} dt + \delta \sigma(t, S_t^0) S_t^0 dW_t^S & \delta S_{t=0}^{LV} &= 0 \\
    d\delta S_t^{SV} &= (r - q)\delta S_t^{SV} dt + \sigma_0 \nu g S_t^0 dW_t^S & \delta S_{t=0}^{SV} &= 0
\end{align*}
\]

- \(\delta S_t^{LV}, \delta S_t^{SV}\) ⇒ order-1 expansion of implied vols \(\hat{\sigma}_{KT}\) and ATMF skew

\[
S_T = \left. \frac{d\hat{\sigma}_{KT}}{d \ln K} \right|_{F_T}
\]

\[
\begin{align*}
    \hat{\sigma}_{KT} &= \hat{\sigma}_T + \delta \hat{\sigma}_{KT}^{LV} + \delta \hat{\sigma}_{KT}^{SV} \\
    S_T &= S_T^{LV} + S_T^{SV}
\end{align*}
\]

- \(\delta \hat{\sigma}_{KT}^{LV}\) and \(\delta \hat{\sigma}_{KT}^{SV}\) given by (cf this morning’s talk + J. Guyon & LB’s paper):

\[
\begin{align*}
    S_T^{LV} &= \frac{1}{T} \int_0^T t \frac{T}{\alpha(t)} dt \\
    S_T^{SV} &= \nu \alpha \theta \left[ (1 - \theta) \rho_{SX1} \frac{k_1 T - (1 - e^{-k_1 T})}{(k_1 T)^2} + \theta \rho_{SX2} \frac{k_2 T - (1 - e^{-k_2 T})}{(k_2 T)^2} \right]
\end{align*}
\]

- No dependence on \(\sigma_0\).

- See appendix for formulae with expansion around \(\sigma(t)\) rather than cst \(\sigma_0\).
Dynamics of ATMF volatilities $\hat{\sigma}_T \equiv \hat{\sigma}_{FTT}$

$\triangleright \; d\hat{\sigma}_T = d\hat{\sigma}_T^{LV} + d\hat{\sigma}_T^{SV}$

$\triangleright \; d\hat{\sigma}_T^{LV}$ generated by LV component:

\[ d\hat{\sigma}_T^{LV} = \frac{d\hat{\sigma}_T}{d \ln S} \, d \ln S \]

$\triangleright \; \frac{d\hat{\sigma}_T^{LV}}{d \ln S}$ in LV given at order 1 by (cf this morning’s talk):

\[ \frac{d\hat{\sigma}_T^{LV}}{d \ln S} = S_T^{LV} + \frac{1}{T} \int_0^T S_t^{LV} \, dt \]

thus:

\[ d\hat{\sigma}_T^{LV} = \left( S_T^{LV} + \frac{1}{T} \int_0^T S_t^{LV} \, dt \right) \sigma_0 dW^S \]

$\triangleright \; \text{What about } d\hat{\sigma}_T^{SV}?$

\[ d\hat{\sigma}_T^{SV} = \nu \alpha \hat{\sigma}_T \left[ (1 - \theta) \frac{1 - e^{-k_1 T}}{k_1 T} \, dW^1 + \theta \frac{1 - e^{-k_2 T}}{k_2 T} \, dW^2 \right] \]
Dynamics of ATMF vols – 2

▶ Bringing everything together:

\[
\frac{d\tilde{\sigma}_T}{\tilde{\sigma}_T} = \left( S_{LV}^T + \frac{1}{T} \int_0^T S_t^{LV} \, dt \right) dW^S + \nu \alpha \theta \left[ (1 - \theta) \frac{1 - e^{-k_1 T}}{k_1 T} dW^1 + \theta \frac{1 - e^{-k_2 T}}{k_2 T} dW^2 \right]
\]

▶ Pbm: only have market smile – don’t know LV function – no \( \alpha(t) \) / no \( S_{LV}^T \).

▶ Only have Market ATMF skew \( S_T \) and \( S_T^{SV} \).

▶ Obtain \( S_{LV}^T \) by subtraction:

\[
S_{LV}^T = S_T^{Mkt} - S_T^{SV}
\]

▶ Expression of \( d\tilde{\sigma}_T \) that uses known inputs:

\[
\frac{d\tilde{\sigma}_T}{\tilde{\sigma}_T} = \left[ \left( S_{Mkt}^T - S_T^{SV} \right) + \frac{1}{T} \int_0^T \left( S_t^{Mkt} - S_t^{SV} \right) \, dt \right] dW^S
\]

\[
+ \nu \alpha \theta \left[ (1 - \theta) \frac{1 - e^{-k_1 T}}{k_1 T} dW^1 + \theta \frac{1 - e^{-k_2 T}}{k_2 T} dW^2 \right]
\]

▶ Now can compute all quantities of interest.

▶ Expressions with expansion around \( \sigma(t) \) given in appendix.
Dynamics of ATMF vols – 3

- Spot/ATMF vol covariance best expressed using SSR:

\[ \mathcal{R}_T = \frac{1}{S_T} \frac{\langle d\sigma_T d\ln S \rangle}{\langle (d\ln S)^2 \rangle} = \frac{\mathcal{R}_T^{LV} S_T^{LV} + \mathcal{R}_T^{SV} S_T^{SV}}{S_T^{LV} + S_T^{SV}} \]

- SSR of mixed model a mixture of LV and SV SSRs weighted by contributions to ATMF skew.

- Little manipulation yields expression for SSR of mixed model:

\[ \mathcal{R}_T = \mathcal{R}_T^{LV} (mkt) + \frac{S_T^{SV}}{S_T} \left[ \mathcal{R}_T^{SV} - \mathcal{R}_T^{LV} (SV) \right] \]

in terms of (a) Mkt smile, (b) SV params.

- Formulae of \( \mathcal{R}_T^{LV} (mkt), S_T^{SV}, \mathcal{R}_T^{SV}, \mathcal{R}_T^{LV} (SV) \) on following slide.

- In numerical tests have used:
  - expansion around \( \sigma(t) \)
  - real (MC) value of \( S_T^{SV} \) – available numerically in 2F model in real time.
Dynamics of ATMF vols – 4

- \( R^{LV}_{T} (\text{mkt}) \): SSR of LV calibrated to market smile

\[
R^{LV}_{T} (\text{mkt}) = 1 + \frac{1}{T} \int_{0}^{T} \frac{S^{mkt}_{t}}{S^{mkt}_{T}} \, dt
\]

- \( S^{SV}_{T} \): ATMF skew of SV model:

\[
S^{SV}_{T} = \nu_{\alpha \theta} \left[ (1 - \theta) \rho_{SX} \frac{k_{1} T - (1 - e^{-k_{1} T})}{(k_{1} T)^{2}} + \theta \rho_{SX} \frac{k_{2} T - (1 - e^{-k_{2} T})}{(k_{2} T)^{2}} \right]
\]

- \( R^{SV}_{T} \): SSR of SV model:

\[
R^{SV}_{T} = \frac{(1 - \theta) \rho_{SX} \frac{1-e^{-k_{1} T}}{k_{1} T} + \theta \rho_{SX} \frac{1-e^{-k_{2} T}}{k_{2} T}}{(1 - \theta) \rho_{SX} \frac{k_{1} T-(1-e^{-k_{1} T})}{(k_{1} T)^{2}} + \theta \rho_{SX} \frac{k_{2} T-(1-e^{-k_{2} T})}{(k_{2} T)^{2}}}
\]

- \( R^{LV}_{T} (SV) \): SSR of LV calibrated to smile generated by the SV portion of the mixed model:

\[
R^{LV}_{T} (SV) = 1 + \frac{1}{T} \int_{0}^{T} \frac{S^{SV}_{t}}{S^{SV}_{T}} \, dt
\]
Example

- Pick as mkt spile smile generated by two-factor model. Parameters typical of Eurostoxx50 smile. VS vols flat at 20%.
  - So that full SV situation attainable.

- Parameters so that $\text{vol}(\hat{\sigma}_T) \propto \frac{1}{T^{0.6}}$.

- $\rho_{SX1}, \rho_{SX2}$ (calibrated on actual smile) so that $S_T \simeq \frac{1}{T^{0.5}}$.

Model params

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu$</td>
<td>310.0%</td>
</tr>
<tr>
<td>$\theta$</td>
<td>13.9%</td>
</tr>
<tr>
<td>$k_1$</td>
<td>8.59</td>
</tr>
<tr>
<td>$k_2$</td>
<td>0.47</td>
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<tr>
<td>$\rho_{XY}$</td>
<td>0%</td>
</tr>
<tr>
<td>$\rho_{SX}$</td>
<td>-54.0%</td>
</tr>
<tr>
<td>$\rho_{SY}$</td>
<td>-62.3%</td>
</tr>
</tbody>
</table>

95/105 vol pts

Model

Power-law exp = 0.5

Mat - years
Example – 2

- **Test 1**: use same parameters for underlying SV model – local vol flat = 1.
  - MC: computed numerically – other curves: order-1 formulae
  - Everything as function of maturity (years).

- **Test 2**: Now halve vol of vol of underlying SV model
How to reconcile different SSRs with same smile?

- Representation of European option prices in diffusive models – see forthcoming book.

\[ P = P_{BS}(0, S_0, \hat{\sigma}_{T,0}^2) + E_{Model} \left[ \int_0^T e^{-rt} \left( \frac{d^2 P_{BS}}{dS d\hat{\sigma}_T^2} \left\langle dS_t d\hat{\sigma}_T^2, t \right\rangle + \frac{1}{2} \frac{d^2 P_{BS}}{(d\hat{\sigma}_T^2)^2} \left\langle d\hat{\sigma}_T^2, t d\hat{\sigma}_T^2, t \right\rangle \right) \right] \]

\( \hat{\sigma}_{T,t} \): VS vol.

- Order 1 in vol of vol + assumption that \( \langle d\ln S_t d\hat{\sigma}_T^2, t \rangle \) indpdt on \( S_t \):

\[ S_T = \frac{1}{2\hat{\sigma}_T^3} \int_0^T \frac{T - t}{T} \langle d\ln S_t d\hat{\sigma}_T^2, t \rangle_0 \]

- Vanilla smile sets spot/ATMF vol integrated covariance "budget" – distribution is model-dependent.

  - In LV, covariance is function of \( t \), concentrated near \( t = 0 \) ⇒ larger SSRs.
    Less covar left for future ⇒ weaker future skews.

  - SV homogeneous model – covar independent on \( t \).
    Future skews ≈ identical to today's skew.

- If \( \langle d\ln S_t d\hat{\sigma}_T^2, t \rangle \equiv f(T - t) \) ⇒ \( f \) fully determined by \( S_{T \in [0, T]} \) ⇒ SSR function of market smile.

\[ S_T \propto \frac{1}{T^\gamma} \Rightarrow \lim_{T \to \infty} R_T = 2 - \gamma. \]
Conclusion

- Characterization of local/stoch vol models that can be used for trading. Pricing function $P(t, S, \hat{\sigma}_{KT}, \lambda)$ has to be such that:

$$\left. \frac{dP}{d\lambda} \right|_{S, \hat{\sigma}_{KT}} = 0$$

Models not obeying this condition $\Rightarrow$ P&L leakage.

- Models obeying condition are genuine market models: thetas matching asset/asset cross-gammas with positive break-even covariance matrix.

- Delta and vega given simply by $\left. \frac{dP}{dS} \right|_{\hat{\sigma}_{KT}}$ and $\left. \frac{dP}{d\hat{\sigma}_{KT}} \right|_{S}$ – the LV function is recalibrated.

- Good approximate expressions for break-even covariances for ATMF vols & spot.
Appendix A - order-one expressions in expansion around $\sigma(t)$

- **LV**: Take
  
  $$\sigma(t, S) = \sigma(t) + \delta\sigma(t, S)$$

  with
  
  $$\delta\sigma(t, S) = \alpha(t) \times, \ x = \ln \left( \frac{S}{F_t} \right)$$

- **$S_L^{LV}, S_V^{SV}$** given by:
  
  $$S_L^{LV} = \frac{1}{T} \int_0^T \frac{\hat{\sigma}_T t \sigma(t)}{\hat{\sigma}_T^2 T} \alpha(t) dt$$

  $$S_V^{SV} = \frac{\nu \alpha \theta}{\hat{\sigma}_T^3 T^2} \int_0^T dt \sigma(t) \int_t^T du \sigma^2(u) \left[ (1 - \theta) \rho_{S_X} e^{-k_1(u-t)} + \theta \rho_{S_X} e^{-k_2(u-t)} \right]$$

  where $\hat{\sigma}_T = \sqrt{\frac{1}{T} \int_0^T \sigma^2(u) du}$.

- Expression of $d\hat{\sigma}_T$ is:
  
  $$d\hat{\sigma}_T = \sigma(0) \left( S_L^{LV} + \frac{1}{T} \int_0^T \sigma_0^2(t) S_L^{LV} dt \right) dW^S$$

  $$+ \nu \alpha \theta \hat{\sigma}_T \left[ (1 - \theta) \frac{\int_0^T \sigma^2(t) e^{-k_1 t} dt}{\int_0^T \sigma^2(t) dt} dW^1 + \theta \frac{\int_0^T \sigma^2(t) e^{-k_2 t} dt}{\int_0^T \sigma^2(t) dt} dW^2 \right]$$
Appendix B - numerical computation of SSR, ATMF vol, spot/vol correl in mixed model

- In two-factor model $\hat{\sigma}_{KT}$ is given by: $\Sigma^M_{KT} (t, S, \sigma, \zeta^u) \equiv \Sigma^M_{KT} (t, S, \sigma, X^1, X^2)$
- Dynamics given by:

$$d\hat{\sigma}_{KT} = \frac{d\Sigma^M_{KT}}{d\ln S} d\ln S + \frac{d\Sigma^M_{KT}}{dX^1} dX^1 + \frac{d\Sigma^M_{KT}}{dX^2} dX^2$$

- Once have the 3 derivatives, compute all covariances of interest

- If only interested in spot/vol covariance:

$$\frac{1}{dt} \left< \left( \frac{d\Sigma^M_{KT}}{d\ln S} d\ln S + \frac{d\Sigma^M_{KT}}{dX^1} dX^1 + \frac{d\Sigma^M_{KT}}{dX^2} dX^2 \right) d\ln S \right>$$

$$= \frac{d\Sigma^M_{KT}}{d\ln S} \sigma^2_0 + \frac{d\Sigma^M_{KT}}{dX^1} \rho_{SX^1} \sigma_0 + \frac{d\Sigma^M_{KT}}{dX^2} \rho_{SX^2} \sigma_0$$

where $\sigma_0 = \sigma (0, S_0) \sqrt{\zeta^0}$ is the initial instantaneous volatility.

- Computed numerically as:

$$\frac{\Sigma^M_{KT} (\ln S_0 + \varepsilon \sigma_0, X^1_0 + \varepsilon \rho_{SX^1}, X^2_0 + \varepsilon \rho_{SX^2}) - \Sigma^M_{KT} (\ln S_0, X^1_0, X^2_0)}{\varepsilon}$$