Modeling VIX futures & VIX ETFs/ETNs

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Outline

- VIX futures, options, ETFs/ETNs, the VXX
- Modeling VIX futures & options
- Connection of VXX smiles to VIX smiles
- Impact of vol-of-vol smile on index smile
- Conclusion

Papers Smile Dynamics I, II, III, IV are available on SSRN website – present material also part of forthcoming book

VIX futures

- VIX futures
  - Expire 30 days before monthly expiries of listed S&P500 options (3d Friday of each month)
  - Settlement value is 30-day LogSwap volatility, as derived from S&P500 option prices ($f_T(S_t)$ forward of S&P500)

$$E\left[-2\ln\frac{S_T}{f_T(S_t)}\right] = (T-t)\,\widehat{\sigma}_T^2(t) \quad \Rightarrow \quad \widehat{\sigma}_T^2(t) = \frac{e^{r(T-t)}}{T-t}\,\mathcal{P}_{\mathrm{mkt},t}^{T}\left[-2\ln\frac{S_T}{f_T(S_t)}\right]$$

$$\widehat{\sigma}_{T_i+\Delta}^2(T_i) = \frac{2e^{r\Delta}}{\Delta}\left(\int_0^{f_{T_i+\Delta}}\mathcal{P}_{\mathrm{mkt},T_i}^{K,T_i+\Delta}\frac{dK}{K^2} + \int_{f_{T_i+\Delta}}^{\infty}\mathcal{C}_{\mathrm{mkt},T_i}^{K,T_i+\Delta}\frac{dK}{K^2}\right)$$

Settlement value of VIX future $F_{T_i}^i$ defined by – drop factor 100 in sequel:

$$F_{t=T_i}^i = 100 * \widehat{\sigma}_{T_i+\Delta}(T_i)$$

  - In terms of forward variances

$$\xi_t^T = \frac{d}{dT}\left((T-t)\,\widehat{\sigma}_T^2(t)\right) \qquad \widehat{\sigma}_T^2(t) = \frac{1}{T-t}\int_t^T \xi_t^\tau\,d\tau$$

$$F_{t=T_i}^i = \sqrt{\frac{1}{\Delta}\int_{T_i}^{T_i+\Delta}\xi_{T_i}^\tau\,d\tau} \qquad F_{t\le T_i}^i = E_t\left[\sqrt{\frac{1}{\Delta}\int_{T_i}^{T_i+\Delta}\xi_{T_i}^\tau\,d\tau}\right]$$

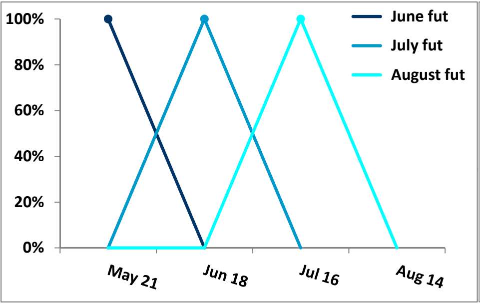
- VIX options trade as well – same maturities as futures – very liquid as well
- ETFs/ETNs on VIX futures very popular
  - either leveraged on 1st nearby future (ex. TVIX)
  - or invested in basket of multiple futures

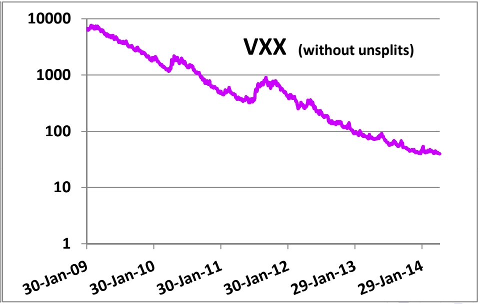
$$\frac{dX_t}{X_t} = (r-q)dt + \sum_{i,\ T_i>t} w_t^i \frac{dF_t^i}{F_t^i}$$

  - VXX: ETN invested in first two nearby futures:

$$\frac{dX_t}{X_t} = (r-q)\,dt + \left[w(T_{1\text{st}}-t)\frac{dF_t^{1\text{st}}}{F_t^{1\text{st}}} + w(T_{2\text{nd}}-t)\frac{dF_t^{2\text{nd}}}{F_t^{2\text{nd}}}\right]$$

$$\begin{cases} w(\tau) = \frac{\tau}{\Delta} & \tau \in [0,\Delta] \\ w(\tau) = 1-\frac{\tau}{\Delta} & \tau \in [\Delta, 2\Delta] \\ w(\tau) = 0 & \tau > 2\Delta \end{cases}$$





- In actual definition of VXX, number of futures – rather than notional – is proportional to $w(\tau)$ – small effect[1]
- VXX options liquid too
- Question: can we price VXX option off VIX smiles? Is there an arbitrage?
- Could model each future independently: ad-hoc multi-asset equity model
- Preferable to use one stoch vol model for VIX futures, VXX, options on variance, S&P500, possibly with adjustable parametrization

[1] See prospectus of VXX ETN at *www.ipathetn.com/static/pdf/vix-prospectus.pdf*

Model

- 2-factor stoch vol model (SD II) – fwd variances have no drift

$$\begin{aligned}\frac{d\xi_t^T}{\xi_t^T} &= (2\nu\alpha_\theta)\left((1-\theta)e^{-k_1(T-t)}dW_t^1 + \theta e^{-k_2(T-t)}dW_t^2\right)\\ \alpha_\theta &= 1/\sqrt{(1-\theta)^2+\theta^2+2\rho_{12}\theta\,(1-\theta)}\end{aligned}$$

- Exponential form for each volatility component allows for Markov representation: ξ_t^T function of Gaussian process x_t^T:

$$\begin{aligned}\xi_t^T \;\equiv\; \xi^T(t,x_t^T) &= \xi_0^T \exp\left((2\nu\alpha_\theta)\,x_t^T - \frac{(2\nu\alpha_\theta)^2}{2}E[(x_t^T)^2]\right)\\ x_t^T &= (1-\theta)\,e^{-k_1(T-t)}X_t + \theta e^{-k_2(T-t)}Y_t\end{aligned}$$

X_t, Y_t Ornstein-Ühlenbeck processes:

$$dX_t = -k_1X_tdt + dW_t^1, \qquad dY_t = -k_2Y_tdt + dW_t^2$$

- Model is auto-calibrated on term structure of LogSwap implied vols; only need to simulate 2 OU processes (+ spot process $dS_t = (r-q)\,S_tdt + \sqrt{\xi_t^t}S_tdW_t^S$)
- How do we parametrize such a model?

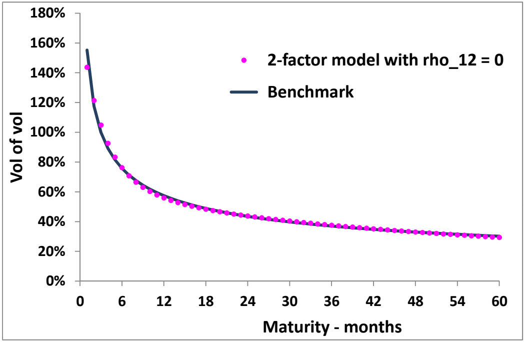
▷ Step I: set parameters $k_1, k_2, \theta, \rho_{12}, \nu$ that determine term structure of volatilities of volatilites.

- Typical benchmark: $\text{vol}(\widehat{\sigma}_T) \;=\; \nu_0 \left(\frac{\tau_0}{T}\right)^{\alpha}$ with $\alpha \in [0.3, 0.6]$

- In two-factor model, for a flat term structure of VS volatilities:

$$\begin{aligned} \text{vol}\left(\widehat{\sigma}_T\right) &= (\nu\alpha_\theta)\sqrt{(1-\theta)^2 I(k_1 T)^2 + \theta^2 I(k_2 T)^2 + 2\rho_{12}\theta\,(1-\theta) I(k_1 T) I(k_2 T)} \\ I\left(x\right) &= \left(1-e^{-x}\right)/x \end{aligned}$$

- Example: take $\tau_0 = 3$months, $\nu_0 = 100\%$, $\alpha = 0.4$. Benchmark vol-of-vol term structure can be captured with various sets:



Some sets with alpha = 0.4		
nu	120.9%	135.8%
theta	57.9%	30.1%
k1	0.58	2.59
k2	1.19	0.32
rho_12	-95%	-50%
nu	174.0%	178.2%
theta	24.5%	23.8%
k1	5.35	6.02
k2	0.28	0.27
rho_12	0%	20%
nu	185.1%	190.1%
theta	23.1%	22.8%
k1	7.26	8.34
k2	0.24	0.22
rho_12	60%	99%

▷ Step II: set spot/factor correlations ρ_{S1}, ρ_{S2} so that

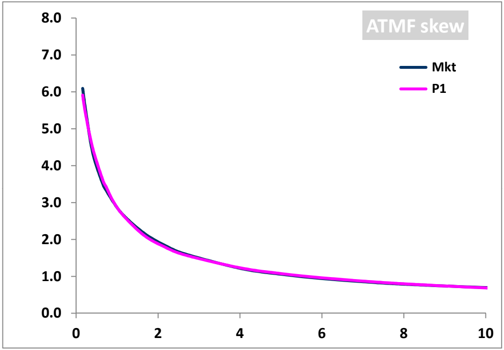
- desired term structure of ATMF skew is obtained,
- or price spot/volatility cross-gammas at desired levels

- In 2-factor model, at order 1 in vol-of-vol, ATMF skew given by:

$$\mathcal{S}_T = \left.\frac{d\widehat{\sigma}_{KT}}{d\ln K}\right|_{\text{ATMF}} = \nu\alpha_\theta\left[(1-\theta)\,\rho_{S1}\frac{k_1T-\left(1-e^{-k_1T}\right)}{(k_1T)^2}+\theta\rho_{S2}\frac{k_2T-\left(1-e^{-k_2T}\right)}{(k_2T)^2}\right]$$

- With 2 factors able to mimic power-law decay of $\mathcal{S}_T \propto \frac{1}{T^\gamma}$. Typically $\gamma \simeq \frac{1}{2}$
- Example: SP500 smile – param set with $\rho_{12} = 0$. Take $\rho_{S1} = -78\%, \rho_{S2} = -33\%$
  ATMF skew $=$ 95/105 smile in pts of vol – mats in years

nu	**174.0%**
theta	**24.5%**
k1	**5.35**
k2	**0.28**
rho_12	**0%**
rho_S1	**-78%**
rho_S2	**-33%**

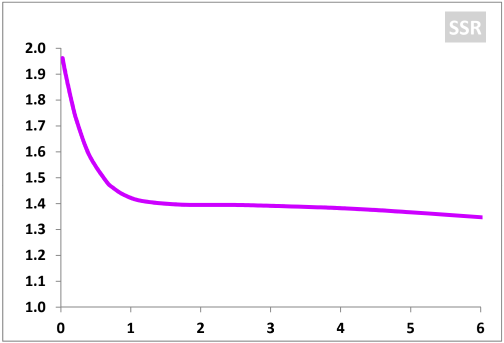


▷ Implied level of Spot/implied vols covariance

- Quantified by SSR – Skew Stickiness Ratio (see SD IV) – given (for general multi-factor case) by:

$$\mathcal{R}_T = \frac{1}{\mathcal{S}_T} \frac{\langle d \ln S \; d\widehat{\sigma}_T \rangle}{\langle d \ln S^2 \rangle} = \frac{\sum_i w_i \; \rho_{Si} \; \frac{1-e^{-k_i T}}{k_i T}}{\sum_i w_i \; \rho_{Si} \; \frac{k_i T - (1-e^{-k_i T})}{(k_i T)^2}}$$

- Example: S&P500 smile – param set in previous slide



- $\lim_{T \to 0} \mathcal{R}_T = 2$

- For power-law decaying ATMF skew $\mathcal{S}_T \propto \frac{T}{T^\gamma}$ then $\mathcal{R} \simeq 2 - \gamma$

VIX futures & options

- VIX future

$$F_t^i = E\left[F_{T_i}^i\left(X_{T_i}, Y_{T_i}\right)\right]$$

$$F_{T_i}^i\left(X_{T_i}, Y_{T_i}\right) = \sqrt{\frac{1}{\Delta}\int_{T_i}^{T_{i+\Delta}} \xi^T(T_i, x_{T_i}^T)dT}$$

$$x_{T_i}^T = (1-\theta)\,e^{-k_1(T-T_i)}X_{T_i} + \theta e^{-k_2(T-T_i)}Y_{T_i}$$

  - F_t^i obtained by 2d integration on X_{T_i}, Y_{T_i}
  - Find better rotated variables by linearizing $\xi^T(T_i, x_{T_i}^T)$:

$$\xi^T(T_i, x_{T_i}^T) \simeq \xi_0^T\left[1 + (2\nu\alpha_\theta)\left[(1-\theta)\,e^{-k_1(T-T_i)}X_{T_i} + \theta e^{-k_2(T-T_i)}Y_{T_i}\right]\right]$$

  - Introduce x:

$$x = \left((1-\theta)\,\frac{1}{\Delta}\int_{T_i}^{T_{i+\Delta}} \xi_0^T e^{-k_1(T-T_i)}dT\right)X_{T_i} + \left(\theta\frac{1}{\Delta}\int_{T_i}^{T_{i+\Delta}} \xi_0^T e^{-k_2(T-T_i)}dT\right)Y_{T_i}$$

  - and $y \perp x$
  - Perform Hermite quadrature on x (a few points) and y (1-2 points)
  - Fast enough that one can price in MC path-dep options on VIX futures

- VIX option

$$P = E\left[\left(F^i_{T_i}(X_{T_i}, Y_{T_i}) - K\right)^+\right]$$

  - Use same rotated variables as before

$$\begin{aligned}
P &= \int_{-\infty}^{+\infty} \rho(y)\,dy \int_{-\infty}^{+\infty} \rho(x) \ \left(F^i_{T_i}(x,y) - K\right)^+ dx \\
&= \int_{-\infty}^{+\infty} \rho(y)\,dy \int_{x^*(y)}^{+\infty} \rho(x) \ \left(F^i_{T_i}(x,y) - K\right) dx \\
&= \int_{-\infty}^{+\infty} \rho(y)\,dy \int_{0}^{+\infty} \rho(x^*(y)+u) \ \left(F^i_{T_i}(x^*(y)+u,y) - K\right) du \\
&= \int_{-\infty}^{+\infty} \rho(y)\,dy \int_{0}^{+\infty} \rho(u)\, e^{-\frac{x^*(y)^2}{2} - x^*(y)u} \ \left(F^i_{T_i}(x^*(y)+u,y) - K\right) du
\end{aligned}$$

where ρ standard normal distribution

  - On y perform std Hermite quadrature with 1-2 points
  - On u, for each value of y, perform special Hermite quadrature scheme for $\int_0^{+\infty} \frac{2}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} f(u)du$ – just a few points

The smile of VIX futures

- So far, in model, inst. fwd variances are lognormal ($\bullet \; = \; 2\nu\alpha_\theta$)

$$\xi^T(t, x_t^T) \; = \; \xi_0^T \; e^{\left(\bullet x_t^T \; - \; \frac{\bullet^2 \overline{(x_t^T)^2}}{2}\right)}$$

- Thus short fwd variances are $\simeq$ lognormal: no good for VIX.
- Also need to capture term structure of VIX implied vols.
  - Step 1: introduce a vol-of-vol rescaling factor ζ_T:

$$\xi^T(t, x_t^T) \; = \; \xi_0^T \; e^{\left((\zeta_T \bullet) x_t^T \; - \; \frac{(\zeta_T \bullet)^2 \overline{(x_t^T)^2}}{2}\right)}$$

  - Step2: upward-sloping smile easily generated by mixing 2 exponentials:

$$\xi^T(t, x_t^T) \; = \; \xi_0^T \left[(1 - \gamma_T) \; e^{\left((\zeta_T \bullet) x_t^T \; - \; \frac{(\zeta_T \bullet)^2 \overline{(x_t^T)^2}}{2}\right)} + \gamma_T e^{\left((\beta_T \zeta_T \bullet) x_t^T \; - \; \frac{(\beta_T \zeta_T \bullet)^2 \overline{(x_t^T)^2}}{2}\right)} \right]$$

  - ζ_T: vol-of-vol rescaling factor, (γ_T, β_T): vol-of-vol smile parameters.
  - For short mats:

$$d\xi^T \; = \; \xi_0^T \zeta_T \left[(1 - \gamma_T) + \gamma_T \beta_T \right] \; (\bullet) dx_t^T$$

  - Renormalization so that (γ_T, β_T) impact *level* of vol of vol minimally:

$$\zeta_T \; \Longrightarrow \; \frac{\zeta_T}{(1 - \gamma_T) + \gamma_T \beta_T}$$

- Adequate in practice – could use general Markov-functional form $\xi_t^T \; = \; \xi_0^T f^T(t, x_t^T)$ – see SD III.

Calibration of VIX market

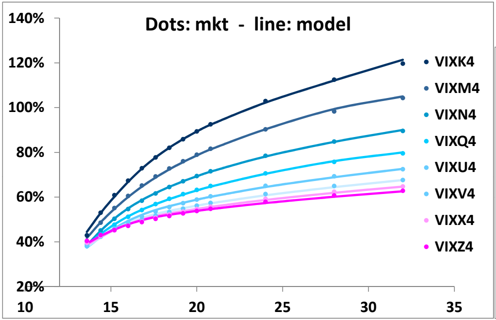
- Procedure N°1
  - Take as input VS or LogSwap vols from S&P500
  - For each VIX future find (ζ, γ, β) so that (a) VIX future is calibrated, (b) VIX option implied vols are matched
- Pbm: VIX market not always consistent with S&P500 VS or LogSwap market prices
  - By combining a position in cash, futures and vanilla options on $F^i_{T_i}$ can create payoff $(F^i_{T_i})^2 = \widehat{\sigma}^2_{T_{i+\Delta}}(T_i)$ – equivalent to a forward LogSwap

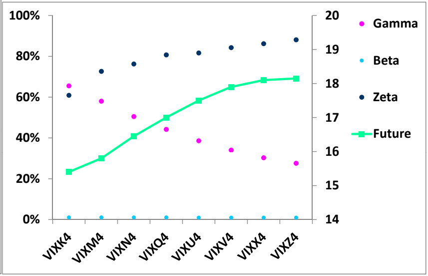
$$
\begin{aligned}
(F^i_{T_i})^2 &= (F^i_t)^2 + 2F^i_t\left(F^i_{T_i} - F^i_t\right) + 2\int_{F^i_t}^{\infty}\left(K - F^i_{T_i}\right)^+ dK + \int_{F^i_t}^{\infty}\left(F^i_{T_i} - K\right)^+ dK \\
\widehat{\sigma}^2_{T_i, T_{i+\Delta}}(t) &= E[(F^i_{T_i})^2] = (F^i_t)^2 + 2\int_{F^i_t}^{\infty}\mathcal{P}^i_K dK + 2\int_{F^i_t}^{\infty}\mathcal{C}^i_K dK
\end{aligned}
$$

  - Generally, LogSwap forward vols derived from VIX market $\geq$ LogSwap (and VS) vols derived from S&P500 smile
- So: procedure N°2
  - For each VIX future F^i generate LogSwap vol over $[T_i, T_{i+\Delta}]$ from VIX future and smile and find triplets $(\zeta_i, \gamma_i, \beta_i)$ so that (a) VIX future and (b) VIX option prices are matched
  - Output: a term-structure of LogSwap vols + triplets $(\zeta_i, \gamma_i, \beta_i)$

- Example: VIX futures & smiles as of April 23, 2014 – use following parameter set

nu	174.0%
theta	24.5%
k1	5.352
k2	0.282
rho_12	0%

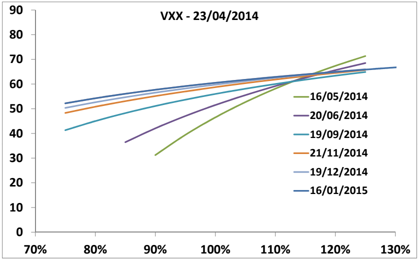


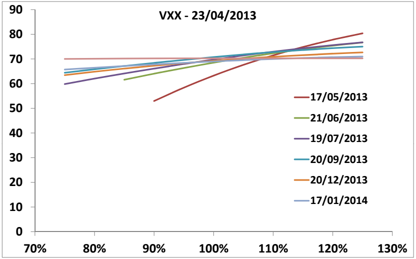


  - Futures exactly calibrated
  - Have used cst value $\beta_i = 1\%$
  - Notice tem structure of VIX implied volatilities

VXX smiles

- VXX smiles as of April 23th 2014 and 2013:[2]





  - Typically much less term structure than in VIX smiles
  - Recall dynamics of VXX:

$$\frac{dX_t}{X_t} = \bullet\, dt + \left(w \frac{dF_t^{\text{1st}}}{F_t^{\text{1st}}} + (1-w)\frac{dF_t^{\text{2nd}}}{F_t^{\text{2nd}}} \right)$$

  - VXX only involves VIX futures that have less than 2 months to expiry
    - VXX options sensitive to correlation of the first two nearby futures
    - VXX options only sensitive to VIX future's variance realized in their last two months
    - VIX options quantify total variance of VIX futures up to their expiries

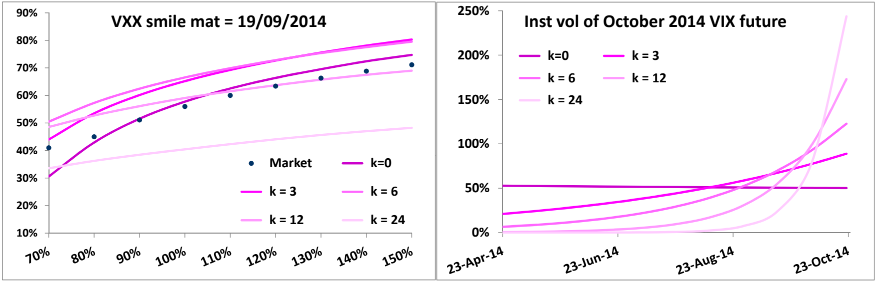
[2] VXX options American, but no dividends and large vols ⇒ in practice European, unless far OTM calls.

Effect of VIX future's volatility distribution

- Experiment: set $\theta = 0$ (➪ one factor model) and vary k_1.
  - Recalibrate VIX futures & smiles as of April 23, 2014 and generate VXX smile
  - Output term structure of instantaneous volatility σ_t of October future:

$$\sigma_t^2 = E_{X_t,Y_t}\left[\left(\frac{dF_t^i(X_t,Y_t)}{F_t^i(X_t,Y_t)}\right)^2\right]$$

    - Computed by easy 2d integration on (X_t, Y_t)
    - $\int_0^{T_i} \sigma_t^2 dt$ is the square of the LogSwap volatility of F_t^i for maturity T_i
    - Squares of curves in right-hand graph all integrate to same value



- VXX near-ATM vols for $k = 0$ and $k = 12$ similar, yet different volatility distribution[3]

[3] Inst vol for $k = 0$ slightly decreasing because of smile.

Effect of correlation

- Recall dynamics of VXX

$$\frac{dX_t}{X_t} = \bullet\, dt + \left(w \frac{dF_t^{1st}}{F_t^{1st}} + (1-w) \frac{dF_t^{2nd}}{F_t^{2nd}} \right)$$

  - Assume for simplicity same volatilities for the two futures:

$$\sigma_X^2 = \sigma^2 \left(w^2 + (1-w)^2 + 2\rho w (1-w) \right) ; \quad w = \frac{T^{1st} - t}{\Delta} \quad \text{for} \quad t \in \left[T^{1st} - \Delta, T^{1st} \right]$$

  - Average over $\left[T^{1st} - \Delta, T^{1st} \right]$: $\sigma_X = \sigma \sqrt{\frac{2+\rho}{3}}$

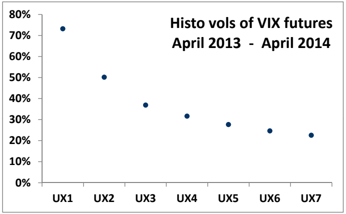
  - Historical correlations of VIX futures (April 2013 - April 2014):

	UX1	UX2	UX3	UX4	UX5	UX6	UX7	SP500
UX1	**100**	**93**	88	87	84	81	77	-80
UX2	**93**	**100**	97	95	92	88	86	-77
UX3	88	97	100	98	96	93	91	-78
UX4	87	95	98	100	98	96	94	-77
UX5	84	92	96	98	100	98	96	-75
UX6	81	88	93	96	98	100	97	-72
UX7	77	86	91	94	96	97	100	-72
SP500	-80	-77	-78	-77	-75	-72	-72	100

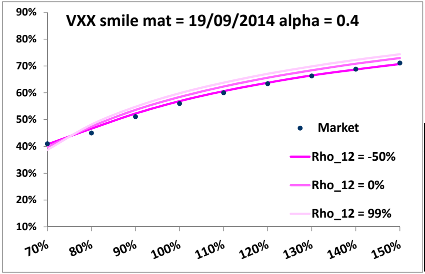
  - Taking $\rho_{\min} = 85\%, \rho_{\max} = 100\%$: ratio of VXX vols $\frac{\sigma_X^{\max}}{\sigma_X^{\min}} = 1.03$
  - For $\sigma_X = 60\%$ impact $\leq$ 2pts of VXX vol. Correlations have small impact

VXX smiles - two-factor model - conclusion

- VXX mostly dependent on vol distribution of VIX futures
  - VXX no redundant instrument – provides info on *implied* vol distribution of VIX futures
  - *Implied* distribution can be different than realized



- With two-factor model: use special sets calibrated on benchmark with for $\sigma_0 = 100\%$, $\tau_0 = 3$ months, $\alpha_0 = 0.4$ for $T \in [0, 9 \text{ months}]$:



Some sets with alpha = 0.4			
nu	227.6%	237.6%	249.4%
theta	32.8%	29.9%	29.5%
k1	10.54	14.43	20.47
k2	1.64	1.23	0.94
rho_12	-50%	0%	99%

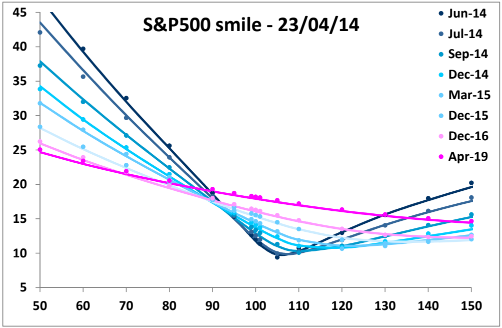
Alternative modeling choice

- In discrete version of fwd variance model (see SD II), use 2-factor model on VIX futures directly – rather than on discrete fwd variances
  - Unsmiled version:
$$\frac{dF_t^i}{F_t^i} = (\nu\alpha_\theta)\left((1-\theta)e^{-k_1(T_i-t)}dW_t^1 + \theta e^{-k_2(T_i-t)}dW_t^2\right)$$
  - Smiled version: $(\bullet = \nu\alpha_\theta)$[4]
$$F_t^i = F_0^i\left[(1-\gamma_i)\, e^{\left((\zeta_i\bullet)x_t^{T_i} - \frac{(\zeta_i\bullet)^2\overline{(x_t^{T_i})^2}}{2}\right)} + \gamma_i e^{\left((\beta_i\zeta_i\bullet)x_t^{T} - \frac{(\beta_i\zeta_i\bullet)^2\overline{(x_t^{T_i})^2}}{2}\right)}\right]$$
$$x_t^{T_i} = (1-\theta)\, e^{-k_1(T-t)}X_t + \theta e^{-k_2(T-t)}Y_t$$
- Supplemented with dynamics of S_t: for $t \in [T_i, T_{i+1}]$
$$dS_t = (r-q)S_t dt + \sigma_i\left(\frac{S_t}{S_{T_i}}, F_{T_i}^i\right)S_t dW_t^S, \qquad \frac{1}{T_{i+1}-T_i}E_{T_i}\left[\int_{T_i}^{T_{i+1}}\sigma_i^2 dt\right] = (F_{T_i}^i)^2$$
- Get control of vol-of-vol smile + decouping of spot / fwd smile – see SD II.

[4]No factor of 2 as, unlike ξ^T, F^i is a volatility.

Back to the S&P500 smile

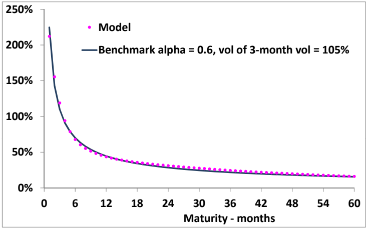
- How is the smile of vol-of-vol manifested in underlying index smile?
  - Take 2-factor model calibrated on S&P500 VS curve – S&P500 smile with $\gamma = \beta = 0$ (no vol-of-vol smile)



nu	300.0%
theta	13.9%
k1	8.59
k2	0.47
rho_12	0%

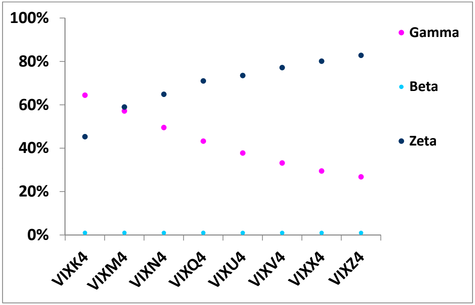
rho_S1	-55.0%
rho_S2	-58.4%

  - Vol of vol params: benchmark with $\alpha = 0.6$ & vol of 3-month vol $= 105\%$



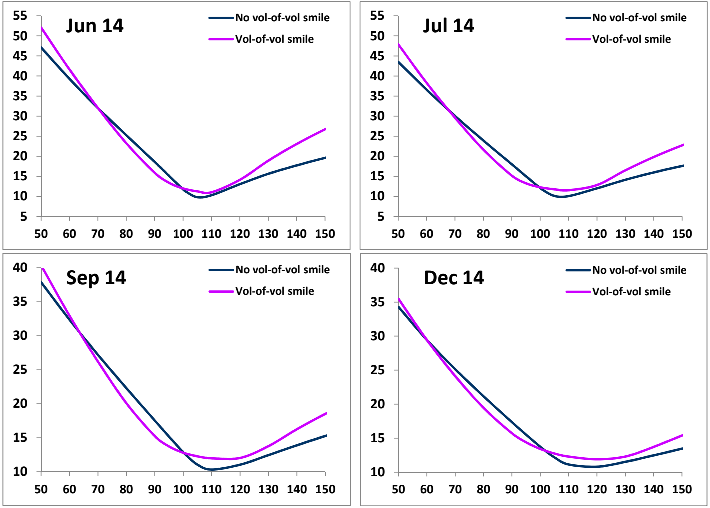
Now with vol of vol smile ...

- Keep params as in previous slide and fit vol-of-vol params on VIX smiles as of 23/04/2014



  - Note that fit on VIX smiles generates $\zeta_i < 100\%$
  - To assess effect of vol-of-vol smile without changing overall level of vol of vol, use (β_i, γ_i) derived from VIX calibration, but keep $\zeta_i = 100\%$

- Now re-generate S&P500 smile



- Smile of vol-of-vol
  - Reduces ATM skew / increases ATM convexity
  - At order 1 in vol of vol, skew independent on vol-of-vol smile
  - 2nd order contribution generates observed behaviour of ATM skew/curvature [5]

[5] See paper *The smile in stochastic volatility models* by J. Guyon & L.B. on SSRN website

General conclusion

- VXX options not redundant – depend on vol distribution of VIX futures
  - Implied vol distribution not stable
- 2-factor forward variance model good workhorse for modeling VIX / variance / S&P500 derivatives
  - Simulation of just two OU processes – no sweat / no time stepping
  - Smile of vol-of-vol comes at no extra cost
  - Provides easy handle on term structure of vols of vols / ATMF skew
  - For decoupling of spot-starting / fwd-starting skew turn to discrete version of same model (see SD II)
- Vol-of-vol levels derived from VIX market can be very different from those we may want to use for S&P500 derivatives
  - Rightly so, since no way to lock vol of vol level by trading S&P500 vanillas
  - Makes zero sense to imply vol of vol from S&P500 smile
  - VIX-derived level of vol-of-vol also different from that implied out of options on realized variance – again no arbitrage
  - VIX smiles very weakly constrained by S&P500 smile (see Pierre's talk)
- Vol-of-vol smile has non-trivial effect on ATM skew.