Smile dynamics: modeling the smile of volatility (of volatility)

Lorenzo Bergomi

Equity Derivatives Quantitative Research
Société Générale
lorenzo.bergomi@sgcib.com

Roma, Global Derivatives 2009

SOCIETE GENERALE
Corporate & Investment Banking

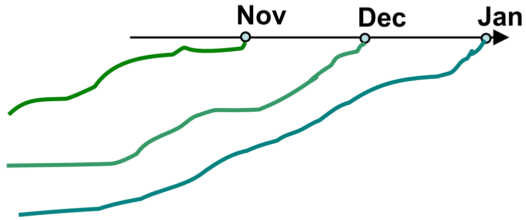
Outline

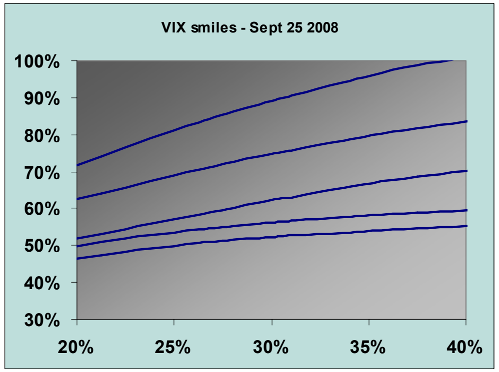
- Rationale for new type of model
- A simple framework for the dynamics of fwd variances
- Generating smiles for fwd variances
- Using the model
    - Calibration to VIX smiles
    - Application to swaptions & options on realized variance
    - The vanilla skew
- The Skew Stickiness Ratio
- Conclusion

SOCIETE GENERALE
Corporate & Investment Banking

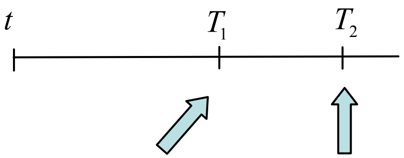
Motivation - 1

- **New listed volatility instruments**
  - VIX futures & options





- **New OTC options on variance**
  - Swaptions
  - Options on realized variance, spot or forward-starting
  - Multi-asset options on realized variance
  - Hybrid equity / realized variance payoffs



$$S = V_{T_1}^{T_1 T_2}$$

$$S = \frac{1}{T_2 - T_1}\sum_{i=T_1}^{T_2}\ln\left(\frac{S_{i+1}}{S_i}\right)^2$$

⇨ **Require modeling of**

- joint dynamics of spot / variance curve
- smile of forward variances

Motivation - 2

- **1-factor stoch. vol models generate poor smile dynamics**
  - One single time scale: for $T >> 1/k$, $\sigma_{\text{Vol}} \propto \frac{1}{T}$, $\text{Skew}_{\text{ATM}} \propto \frac{1}{T}$

- **Popular smile models have other structural problems**
  - Heston model:
    - short ATM vol is normal
    - short ATM skew inv. prop. to level of ATM vol
    - simulation of dynamics of variance curve uselessly arduous
  - Jump / Lévy models:
    - forward skew but no dynamics of vol / skew
    - process for producing skew generates unrealistic skewness for short-term returns
      ⇨ pbs for very path-dependent options
  - Stoch. Vol Lévy
    - dynamics is very constrained: for short maturities $\left.\frac{d\hat{\sigma}}{d\ln K}\right|_F \propto \frac{1}{\hat{\sigma}_F}$

A simple framework - 1

- **Start with simple lognormal dynamics for forward variances driven by OU factors:**

$$\xi_t^T = \xi_0^T e^{\omega x_t^T - \frac{\omega^2}{2} E[(x_t^T)^2]} \qquad \xi_0^T = \frac{d}{dT}\left(T\hat{\sigma}^2_{t=0}(T)\right)$$

$$x_t^T = (1-\theta)e^{-k_1(T-t)}X_t + \theta e^{-k_2(T-t)}Y_t$$

$$dX = -k_1 X dt + dW^X$$

$$dY = -k_2 Y dt + dW^Y$$

X_t, Y_t are Gaussian: correlation of W^X and W^Y is ρ_{XY}

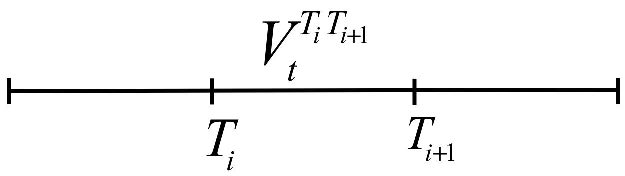
- **Dynamics is Markovian:**
  - **Fwd variances are functions of just 2 easy-to-simulate factors:** $V_t^{T_1T_2}(t, X_t, Y_t) = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \xi_t^T dT$

$$\frac{dV^{T_1T_2}}{V^{T_1T_2}} = \alpha_X^{T_1T_2} dW^X + \alpha_Y^{T_1T_2} dW^Y$$

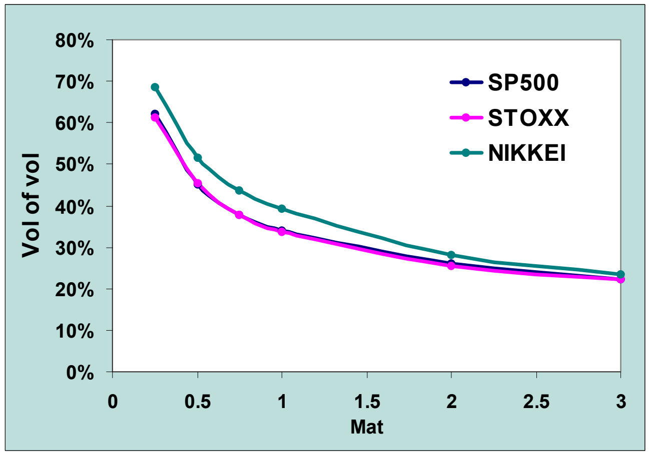
$$\alpha_X^{T_1T_2} = (1-\theta)\frac{\int_{T_1}^{T_2} \xi_t^T e^{-k_1(T-t)} dT}{\int_{T_1}^{T_2} \xi_t^T dT}, \quad \alpha_Y^{T_1T_2} = \theta \frac{\int_{T_1}^{T_2} \xi_t^T e^{-k_2(T-t)} dT}{\int_{T_1}^{T_2} \xi_t^T dT}$$

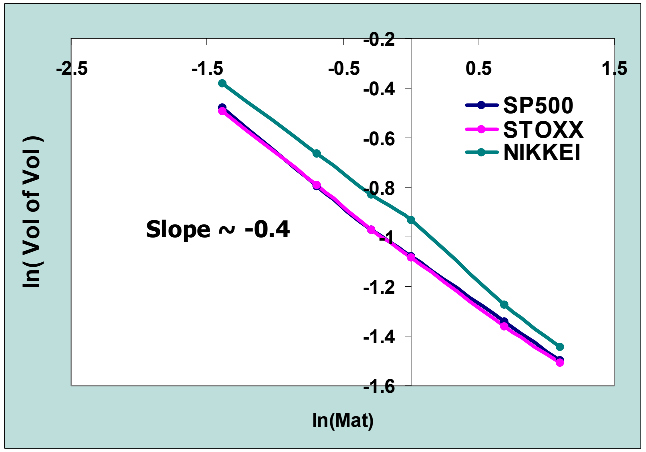
A simple framework - 2

- **We can use same dynamics for discrete forward variances**
  - Allows for independent control of short fwd skew



- **2-factor model is minimal setup that provides handle on:**
  - term-structure of volatilities of volatilities
  - term-structure of skew





- Adding more factors generates no additional complexity:

$$\xi_t^T = \xi_0^T e^{\omega x_t^T - \frac{1}{2}\omega^2 E[(x_t^T)^2]} \qquad x_t^T = \sum_n w_n e^{-k_n (T-t)} X_t^n$$

Smile for fwd variances – continuous 1

- **Consider instantaneous fwd variances ξ^T**
- **Would like to keep model Markovian while relaxing lognormality:**
  - Define

$$x_t^T = \alpha_\theta\left((1-\theta)e^{-k_1(T-t)}X_t + \theta e^{-k_2(T-t)}Y_t\right) \quad \text{with} \quad \alpha_\theta = 1/\sqrt{(1-\theta)^2+\theta^2+2\rho_{XY}\theta(1-\theta)}$$

  - Write $\xi_t^T = \xi_0^T \, f^T(x_t^T, t)$

  - Conditions on f^T:
    - ξ^T is driftless: $\frac{df^T}{dt} + \frac{\lambda^2(T-t)}{2}\frac{d^2 f^T}{dx^2} = 0$

      with

      $$\lambda^2(\tau) = \alpha_\theta^2\left[(1-\theta)^2 e^{-2k_1\tau} + \theta^2 e^{-2k_2\tau} + 2\rho_{XY}\theta(1-\theta)e^{-(k_1+k_2)\tau}\right]$$

    - $f^T(0,0) = 1$
    - f^T monotonic in x

  - Ex: lognormal dynamics given by $f^T(x,t) = e^{\omega x - \frac{\omega^2}{2}\int_{T-t}^{T}\lambda^2(\tau)d\tau}$

Smile for fwd variances – continuous 2

- **Analogous to "Markov-functional" models of fixed income**
  - Is in fact a local vol model:

$$\sigma\left(\xi^T, t\right) = \eta(T-t)\left.\frac{\partial \ln f^T}{\partial x}\right|_{x=f^{T^{-1}}(\xi,t)}$$

  - f^T uniquely determined by terminal profile: $f^T(x, t=T)$

- **No market smiles for individual ξ^T, rather for discrete fwd variances:** $V_t^{T_i T_{i+1}} = \frac{1}{T_{i+1}-T_i}\int_{T_i}^{T_{i+1}} \xi_t^T dT$
  - (1) Assume $f^T(x, T_i)$ are identical for $T \in [T_i, T_{i+1}[$: exact calibration on VIX smiles
  - (2) Use analytical ansatz for f^T : representation on basis of space-time harmonic functions

$$f^T(x,t) = \int_0^{+\infty} d\mu(\omega) e^{\omega x - \frac{\omega^2}{2}\int_{T-t}^{T} \lambda(\tau)^2 d\tau}$$

Using just 2 exponentials:

$$f^T(x,t) = (1-\gamma_T) e^{\omega_T x - \frac{\omega_T^2}{2}\int_{T-t}^{T}\lambda(\tau)^2 d\tau} + \gamma_T e^{(\beta_T \omega_T)x - \frac{(\beta_T\omega_T)^2}{2}\int_{T-t}^{T}\lambda(\tau)^2 d\tau} \quad \text{with} \quad \omega_T = (2\nu\zeta_T)\frac{1}{1-\gamma_T+\gamma_T\beta_T}$$

Initially vol of very short vol is ν: $d\xi_{t=T}^T = (2\nu\zeta^T)\ \xi^T dx^T$

Smile for fwd variances – continuous 3

- **Process for spot**

$$dS = (r-q)Sdt + \sqrt{\xi_0^t}\sqrt{f^t(x_t^t, t)}S\,dW_t$$

$$x_t^t = \alpha_\theta\left((1-\theta)X_t + \theta Y_t\right)$$

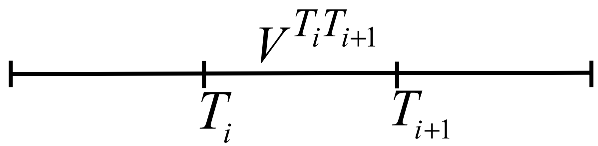
- **Vanilla skew is set by correlations** ρ_{SX}, ρ_{SY}

- **Pricing is painless**
  - Processes X_t, Y_t are simulated (1) exactly, (2) with no time-stepping
  - Brownian motion for spot process

- Before we turn to examples: discrete version

Smile for fwd variances – discrete 1

- **Assume a tenor structure of expiries, for example those of VIX futures & options**



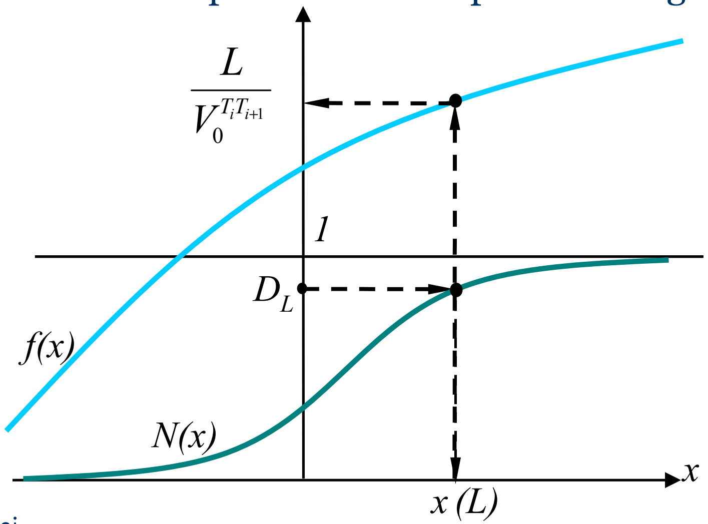
- **Model discrete fwd variances $V^{T_iT_{i+1}}$:**

$$V_t^{T_iT_{i+1}} = V_0^{T_iT_{i+1}} f^i(x_t^i, t)$$

$$x_t^i = \alpha_\theta\left((1-\theta)e^{-k_1(T_i-t)}X_t + \theta e^{-k_2(T_i-t)}Y_t\right)$$

$$\frac{df^i}{dt} + \frac{\lambda^2(T_i-t)}{2}\frac{d^2 f^i}{dx^2} = 0$$

- **Determine terminal profile $f^i(x, t=T_i)$ so as to exactly recover market smile for $V^{T_iT_{i+1}}$**
  - Use procedure outlined by J. Kennedy, P. Hunt, A. Pelsser
  - VIX smiles provide market prices for digitals on $\sqrt{V_{T_i}^{T_iT_{i+1}}}$ hence on $V_{T_i}^{T_iT_{i+1}}$



$$D_L = P(V_{T_i}^{T_iT_{i+1}} < L) = P(x_{T_i}^i < x(L)) \quad \text{with} \quad f^i(x(L), T_i) = \frac{L}{V_0^{T_iT_{i+1}}}$$

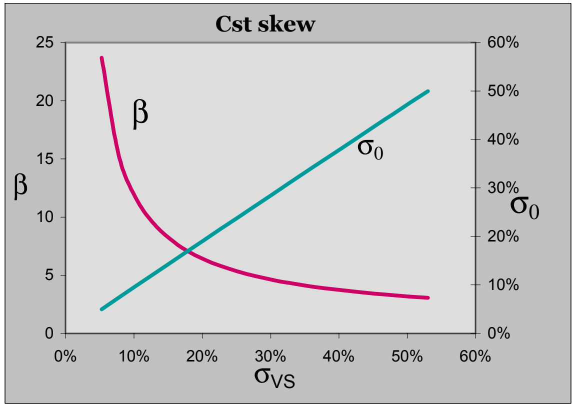
Smile for fwd variances – discrete 2

- **Process for spot over interval $[T_i, T_{i+1}[$**
  - Solution (1): variance over interval $[T_i, T_{i+1}[$ stays constant, equal to $V_{T_i}^{T_i T_{i+1}} = V_0^{T_i T_{i+1}} f^i(x_{T_i}^i, T)$

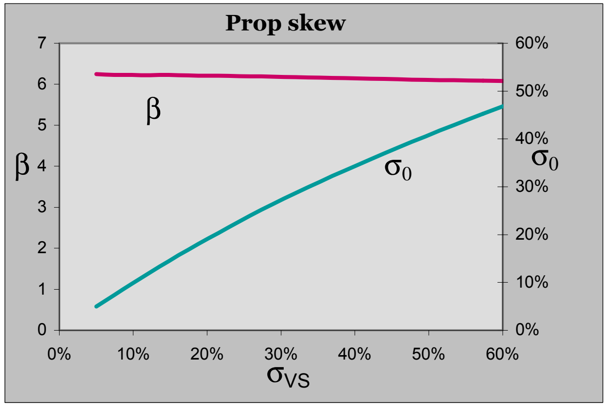
    ⇨ both vanilla skew and fwd skew set by correlations ρ_{SX}, ρ_{SY}

  - Solution (2): control fwd skew by choosing "local vol" $\sigma_i(S) \equiv \varphi\left(\frac{S}{S_{T_i}}, V_{T_i}^{T_i T_{i+1}}\right)$ such that:
    - VS variance for maturity T_{i+1} is $V_{T_i}^{T_i T_{i+1}}$
    - Desired level of fwd skew over $[T_i, T_{i+1}[$ is obtained
    - Desired dependence of level fwd skew on level of vol is achieved
  - For example, use CEV form: $\varphi(x) = \sigma_0 x^{1-\beta}$

$\hat{\sigma}_{95\%} - \hat{\sigma}_{105\%} = 5\%$



$\frac{\hat{\sigma}_{95\%} - \hat{\sigma}_{105\%}}{\hat{\sigma}_{100\%}} = 0.25$



SOCIETE GENERALE
Corporate & Investment Banking

Smile for fwd variances – discrete 3

- **Benefits of using discrete version:**

  - Control on smile of fwd vols – VIX smiles easily calibrated
  - Control on short fwd skew and its dependence on level of fwd vol
  - Correlations ρ_{SX}, ρ_{SY} still at our disposal to control vanilla smile independently

  - Again pricing is painless:
    - Processes X_t, Y_t are simulated (1) exactly, (2) with no time-stepping
    - Brownian motion for spot process
    - Mapping functions $f^i(x, t)$ are generated once and for all

SOCIETE GENERALE
Corporate & Investment Banking

Using the model – calibration to VIX smiles - 1

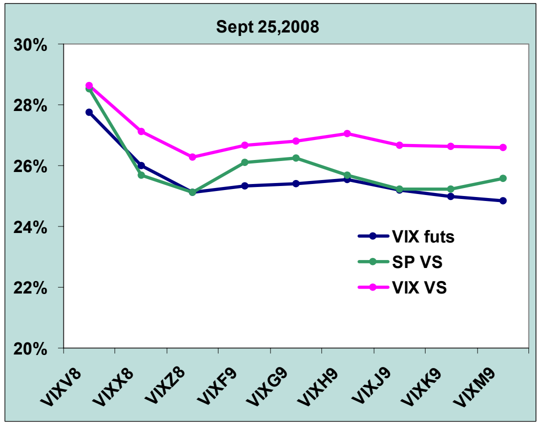
- **Here we use continuous version**
- **Calibration on VIX smiles:**

- Choose values for $\nu, \theta, k_1, k_2, \rho_{XY}$: sets general dynamics of fwd vols in model
- Use piecewise constant parameters $\gamma_i, \beta_i, \zeta_i$ on each interval $[T_i, T_{i+1}[$
  - γ, β control skew / ζ controls vol level – of fwd vol

- Use VS term-structure computed on SP500 smile as input
- Calibrate both
  - VIX futures themselves $F = E[\sqrt{V_{T_i}^{T_i T_{i+1}}}]$
  - VIX options $C_K = E[(\sqrt{V_{T_i}^{T_i T_{i+1}}} - K)^+]$

- Is VIX market consistent with SP500 option's market ?
  - 1-month fwd variance can be replicated with VIX futures & options

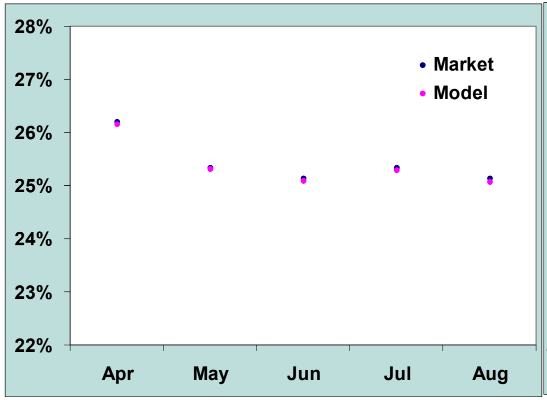
$$V^{T_i T_{i+1}} = F^2 + 2\int_0^F P_K dK + 2\int_F^\infty C_K dK$$

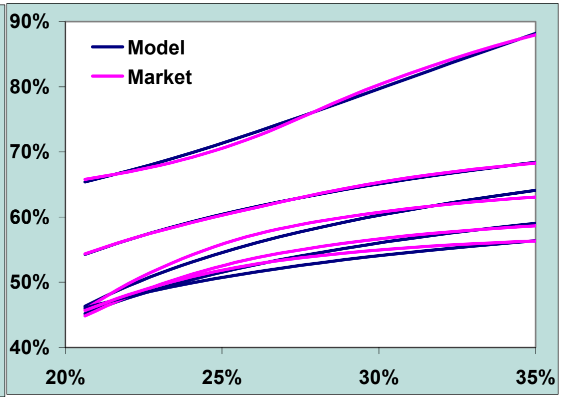


Using the model – calibration to VIX smiles – 2

- Pricing of options on fwd vol/var easy: simple Gaussian integration

•March 18, 2008

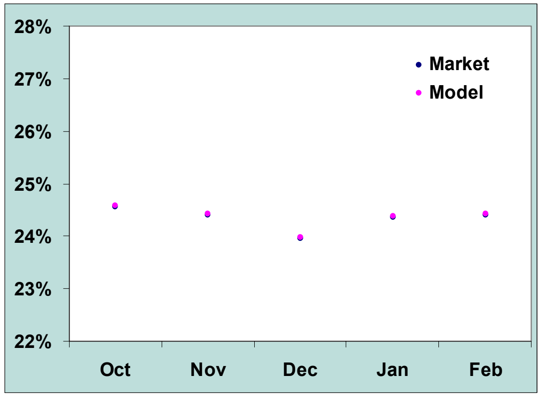


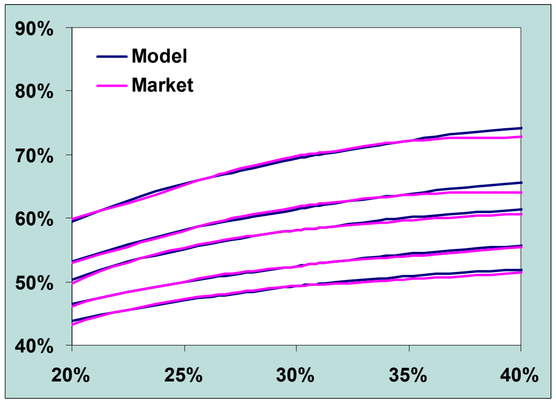


	γ_i	β_i	ζ_i
16-Apr-08	87%	21%	105%
21-May-08	36%	11%	94%
18-Jun-08	35%	0%	95%
16-Jul-08	30%	0%	99%
20-Aug-08	24%	0%	100%

ν	130.0%
θ	28%
k_1	8.0
k_2	0.35
ρ_{XY}	0%

•Sept 12, 2008

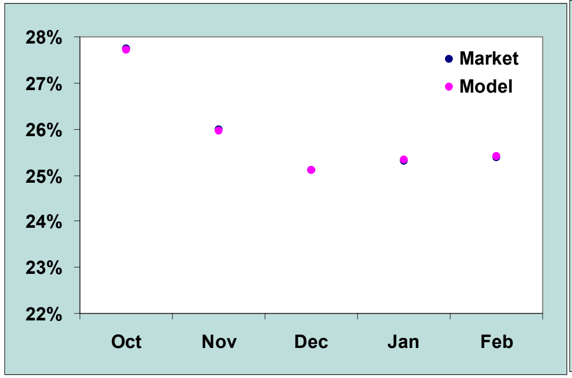


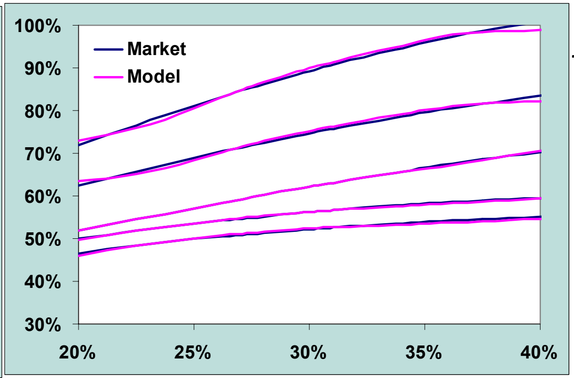


	γ_i	β_i	ζ_i
22-Oct-08	35%	16%	87%
19-Nov-08	26%	5%	89%
17-Dec-08	23%	5%	95%
21-Jan-09	39%	26%	91%
18-Feb-09	23%	9%	90%

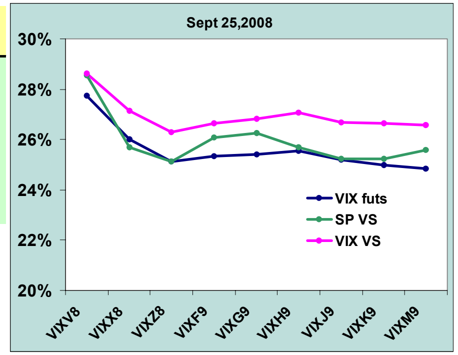
Using the model – calibration to VIX smiles – 3

• Sept 25, 2008

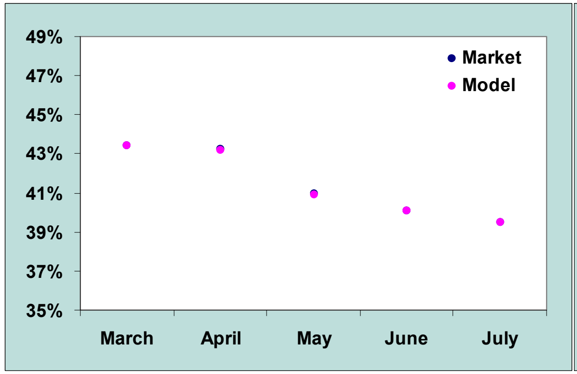


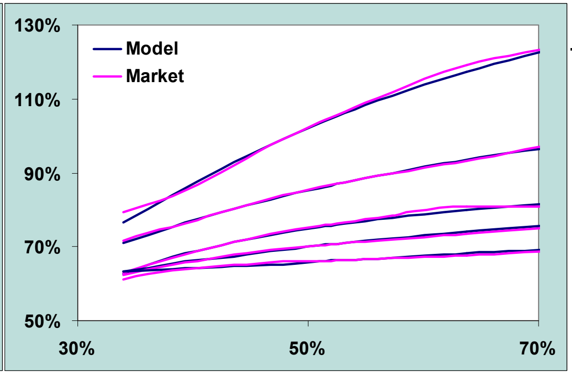


	γ_i	β_i	ζ_i
22-Oct-08	76%	21%	114%
19-Nov-08	70%	28%	109%
17-Dec-08	76%	27%	104%
21-Jan-09	41%	29%	96%
18-Feb-09	30%	21%	94%

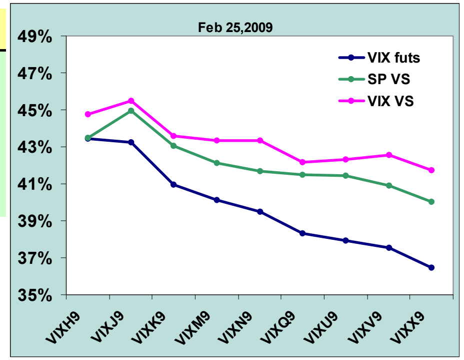


▪ Feb 25, 2009

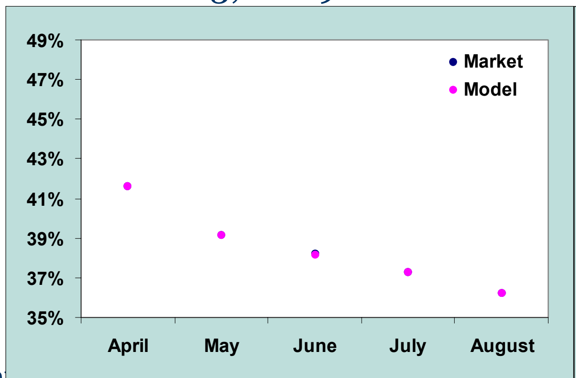


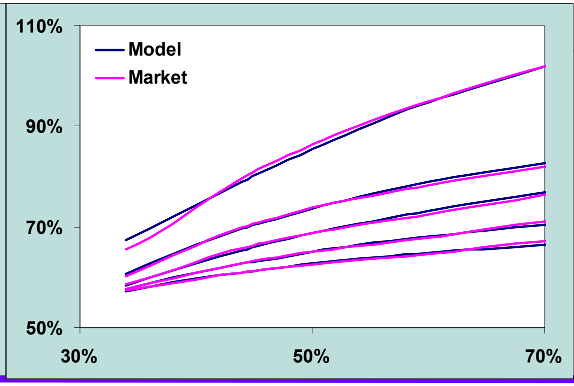


	γ_i	β_i	ζ_i
18-Mar-09	70%	20%	122%
15-Apr-09	59%	26%	123%
20-May-09	26%	0%	119%
17-Jun-09	60%	37%	123%
22-Jul-09	89%	39%	125%

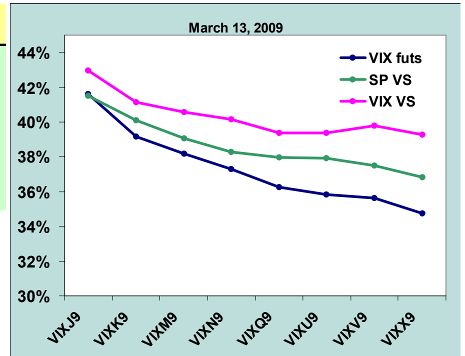


▪ March 13, 2009





	γ_i	β_i	ζ_i
15-Apr-09	76%	23%	109%
20-May-09	54%	21%	109%
17-Jun-09	62%	28%	114%
22-Jul-09	50%	30%	116%
19-Aug-09	40%	31%	116%

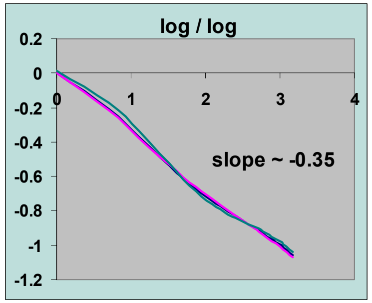


Using the model – options on realized variance – swaptions

- Option on realized variance pays $\left(\frac{1}{T}\sum_0^T \ln\left(\frac{S_{i+1}}{S_i}\right)^2 - K\right)^+$, swaption pays $\left(V_{T_1}^{T_1T_2} - K\right)^+$

- Popular approximation:
  - Define underlying effective underlying Y: $Y = \dfrac{t\sigma_{rt}^2 + (T-t)\hat{\sigma}_{t,T-t}^2}{T}$
  - Y hedged by trading VS of maturity T: Y is driftless
  - Dynamics for Y depends on dynamics of $\hat{\sigma}_{t,T-t}$

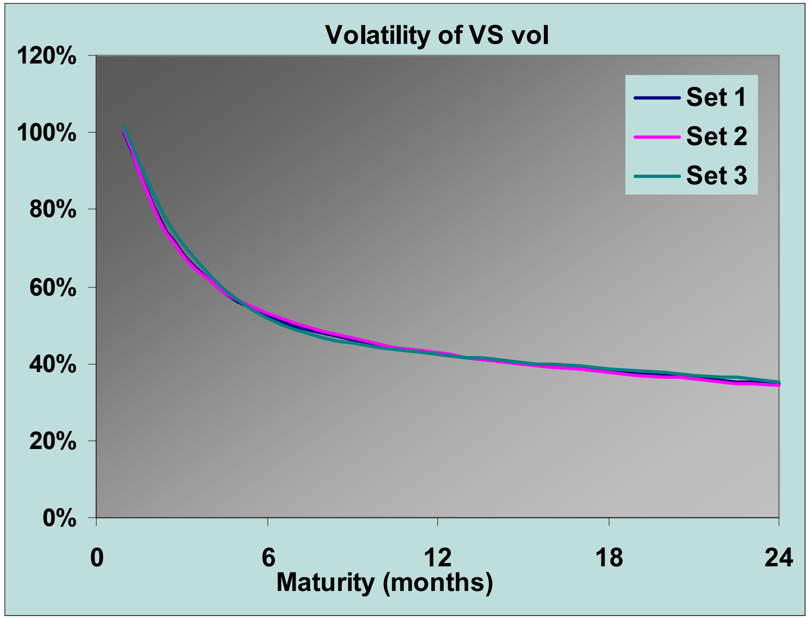
  ⇨ Price of option on variance only depends on dynamics of VS vol of residual maturity.



- Price option with model

	Set1	Set 2	Set 3
ν	130.0%	137.0%	125.0%
θ	28%	29%	32%
k_1	8.0	12.0	4.5
k_2	0.35	0.30	0.60
ρ_{XY}	0%	90%	-70%

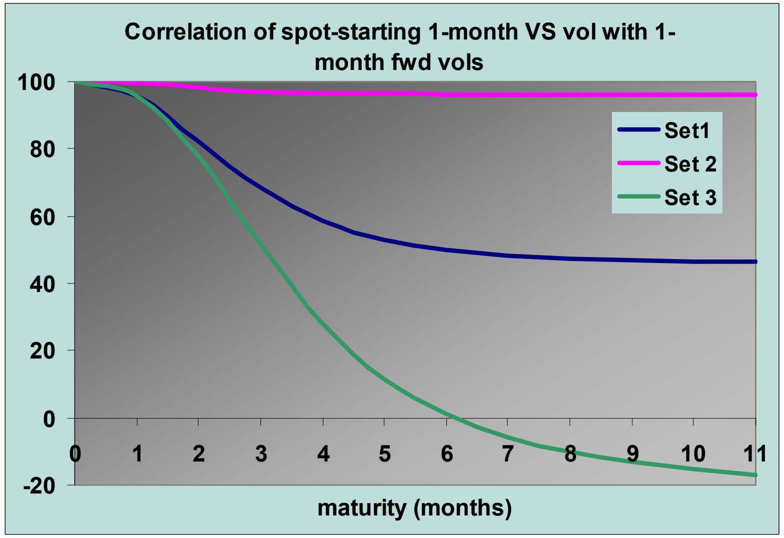
Spot-starting realised				
	Set1	Set 2	Set 3	
6 months	2.03%	2.03%	2.03%	$\left(\frac{1}{T}\sum_0^T \ln\left(\frac{S_{i+1}}{S_i}\right)^2 - K\right)^+$
1 year	2.22%	2.21%	2.24%	
6 months realized in 6 months				$\left(\frac{1}{T_2-T_1}\sum_{T_1}^{T_2} \ln\left(\frac{S_{i+1}}{S_i}\right)^2 - K\right)^+$
	3.04%	2.89%	3.25%	
6 months in 6 months swaption				$\left(V_{T_1}^{T_1T_2} - K\right)^+$
	2.28%	2.10%	2.57%	

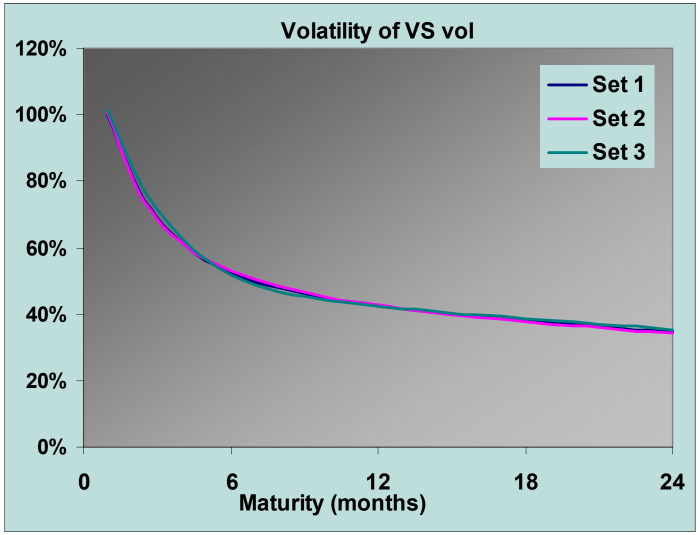


⇨ Pop. approx ok for spot-starting options on realized variance

Using the model – dynamics of fwd vols

	Set1	Set 2	Set 3
ν	130.0%	137.0%	125.0%
θ	28%	29%	32%
k_1	8.0	12.0	4.5
k_2	0.35	0.30	0.60
ρ_{XY}	0%	90%	-70%





⇨ term-structure of vols of spot-starting vols does not uniquely determine vols & correls of fwd vols

Spot-starting variance = basket of fwd variances. Same basket vol can be achieved with:

- low vols / high correls
- high vols / low correls

For flat term-structure of VS vols, inst. vol of VS vol given by:

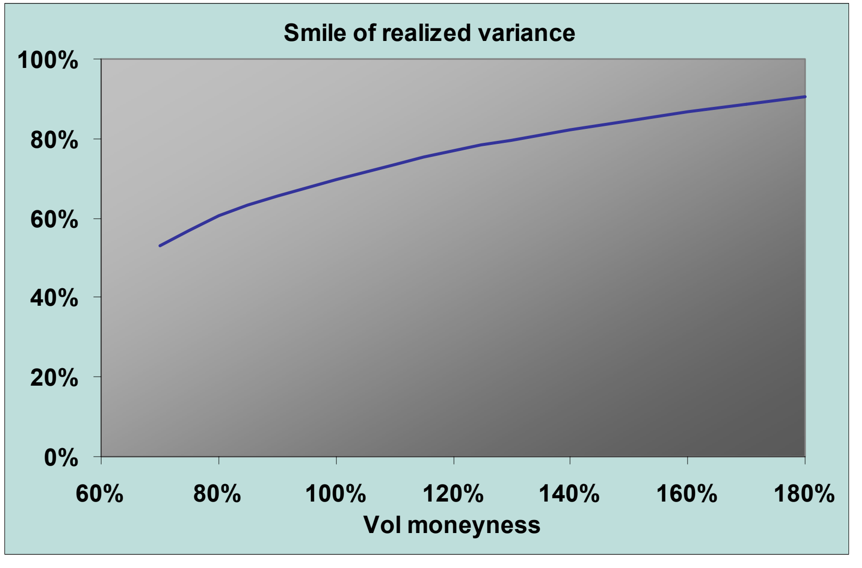
$$\sigma_T^{\text{vol}} = \nu\alpha_\theta\sqrt{\theta^2\left(\frac{1-e^{-k_1T}}{k_1}\right)^2+(1-\theta)^2\left(\frac{1-e^{-k_2T}}{k_2}\right)^2+2\rho_{XY}\theta(1-\theta)\left(\frac{1-e^{-k_1T}}{k_1}\right)\left(\frac{1-e^{-k_2T}}{k_2}\right)}$$

SOCIETE GENERALE
Corporate & Investment Banking

Using the model – options on realized variance – smile

- Use parameters calibrated on VIX smiles of march 18, 2008

- Price option on realized variance – back out smile:



Using the model – the vanilla skew – 1

- Vanilla skew in continuous version is generated by spot / vol correlations ρ_{SX}, ρ_{SY}

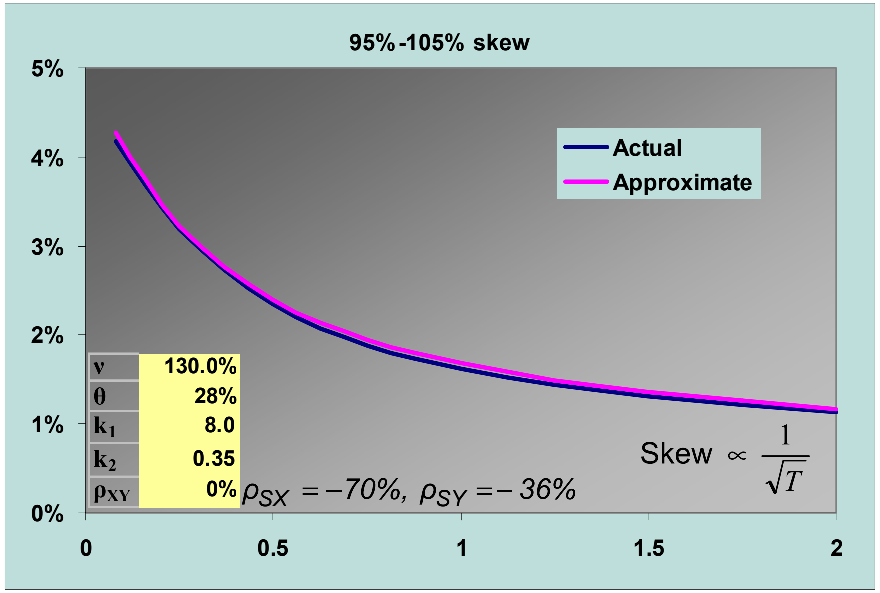
- Approximate expression for vanilla skew:
  - ATM skew is related to skewness Σ_T of $x = \ln\left(\frac{S_T}{S_0}\right)$ through: $\left.\frac{d\hat{\sigma}}{d\ln K}\right|_F = \frac{\Sigma_T}{6\sqrt{T}}$
  - is given at 1st order in vol-of-vol by (double) integral of spot/vol correlation function:

$$\left\langle (x - <x>)^3 \right\rangle = 6\sigma_0 \int_0^T dt \int_0^t \left\langle \frac{dS_\tau}{S_\tau}\, \sigma_t \right\rangle$$

⇨ yields expression of ATMF skew at order 1 in vol of vol:

$$\left.\frac{d\hat{\sigma}}{d\ln K}\right|_F = \nu\alpha_\theta \left[(1-\theta)\rho_{SX} \frac{k_1 T - (1 - e^{-k_1 T})}{k_1^2 T^2} + \theta \rho_{SY} \frac{k_2 T - (1 - e^{-k_2 T})}{k_2^2 T^2} \right]$$

- Example: 95% - 105% skew with $\rho_{SX} = -70\%$, $\rho_{SY} = -36\%$



⇨ **Parameters generating slow decay of vol of vol also generate slow decay of skew**

SOCIETE GENERALE
Corporate & Investment Banking

The Skew Stickiness Ratio – 1

• Natural question : *how does the VS or ATMF vol move when spot moves ?*

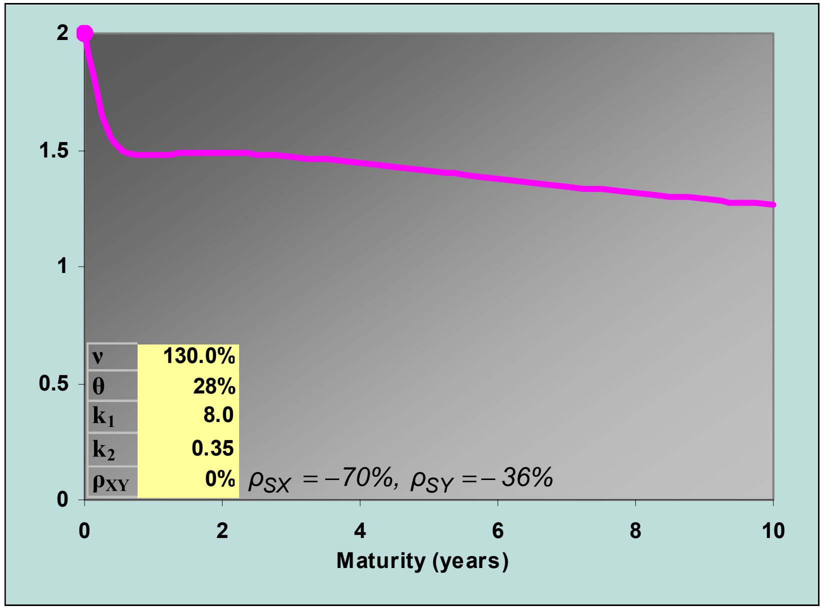
- in units of the ATMF skew ?

⇨ enter the Skew Stickiness Ratio

$$R_T = \frac{\left(\frac{\delta \sigma_{VS}^T}{\delta \ln S}\right)}{\left.\frac{d\hat{\sigma}}{d \ln K}\right|_{F,T}} \quad \text{with} \quad \frac{\delta \sigma_{VS}}{\delta \ln S} = \frac{\langle \delta\sigma_{VS}\, \delta \ln S \rangle}{\langle (\delta \ln S)^2 \rangle}$$

• At 1st order in vol of vol, with flat VS term-structure:

- **For short maturities** $R_T \to 2$ **: model-independent**
- **For (very) long maturities** $R_T \to 1$



• General expression for any number of factors:

$$R_T = \frac{\sum w_i \rho_{iS} \frac{1 - e^{-k_i T}}{k_i T}}{\sum w_i \rho_{iS} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2}}$$

The Skew Stickiness Ratio – 2

- Model-independent (no jumps) expression of the SSR at order 1 in the vol-of-vol (flat term-structure of vols):

$$R_T = \frac{\frac{1}{T}\int_0^T f(t)dt}{\frac{1}{T^2}\int_0^T dt \int_0^t f(\tau)d\tau}$$

where $f(\tau)$ is the spot/vol correlation function: $f(\tau) = \frac{1}{dt}\left\langle \frac{dS_t}{S_t}\sigma_{t+\tau}\right\rangle$

- If f is monotonic then $1 \leq R \leq 2$

- For $T \to 0$ above expression recovers well-known model-independent result that $R \to 2$

The Skew Stickiness Ratio – 3

- Scaling of $f(\tau)$ for large τ governs
  - ➩ **The speed with which the ATMF skew decays**
  - ➩ **The value of the SSR for large T**

- Imagine that for large t $f(\tau) \propto \dfrac{1}{\tau^{\beta}}$ then :

  - If $\beta > 1$ $\quad \left.\dfrac{d\hat{\sigma}}{d\ln K}\right|_F \propto \dfrac{1}{T}$ and $R_T \to 1$
  - If $\beta < 1$ $\quad \left.\dfrac{d\hat{\sigma}}{d\ln K}\right|_F \propto \dfrac{1}{T^{\beta}}$ and $R_T \to 2-\beta$

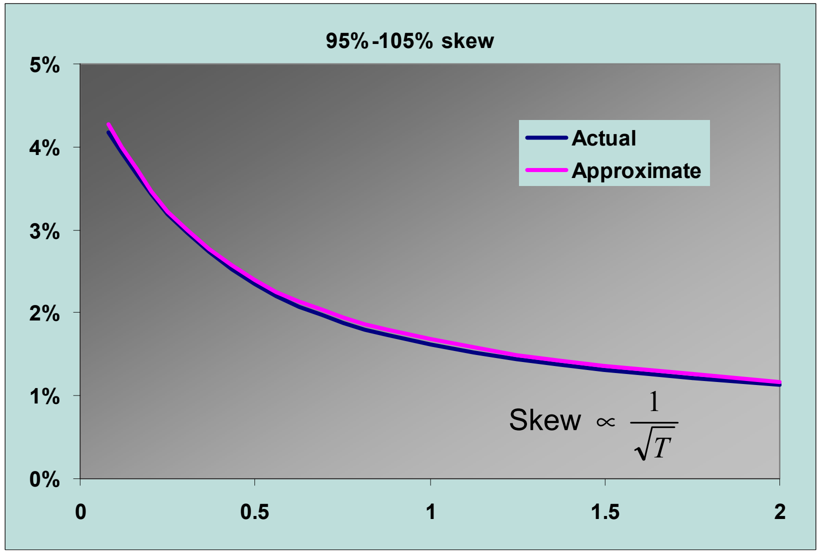
Exponential decay same as $\beta > 1$

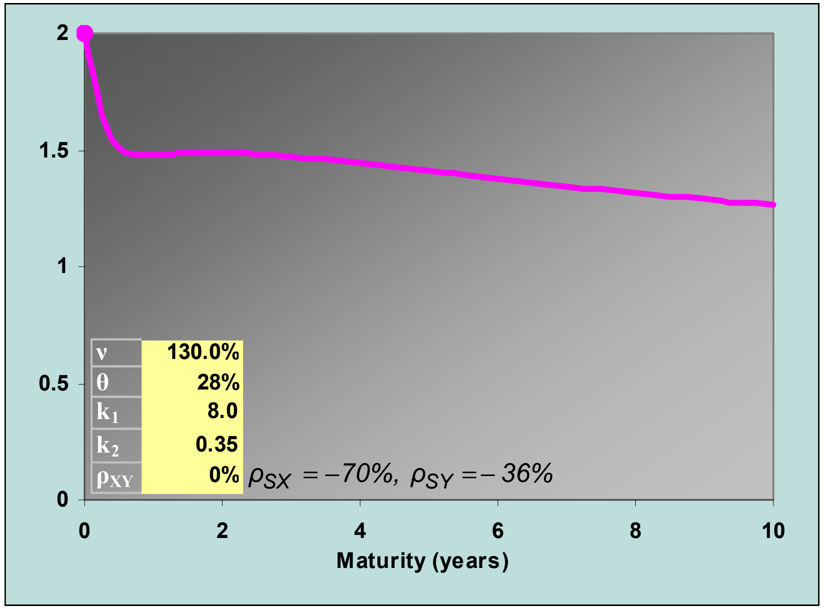
➩ The decay of the ATMF skew (static feature) and the large-T value of the SSR (dynamic feature) are related for large T through:

$$\text{Skew}_T \propto \frac{1}{T^{2-R}}$$

➩ Explains why in our example:

- intermediate regime where Skew$_T \propto 1/T^{0.5}$, $R_T \approx 1.5$
- eventually for long maturities Skew$_T \propto 1/T$, $R_T \approx 1$





SOCIETE GENERALE
Corporate & Investment Banking

Conclusion

- 2-factor stoch vol model allows control over dynamics of fwd vols
- Model is driven by easy-to-simulate Gaussian processes – no sweating
- Can be extended to *N* factors with no additional complexity
  - needed for options on slope of VS curve
- Allows control on smile of fwd vols – almost for free
  - can be calibrated to VIX smiles, exactly in discrete version
- Discrete version allows additional control on short fwd skew and decoupling from vanilla skew
  - magnitude & dependence on level of short fwd vol
- Parameters governing dynamics of fwd vols also govern decay of vanilla skew
- Model-independent relationship at 1st order in the vol-of-vol links SSR to decay of the ATMF skew