Errata – chapter 8

The smile of stochastic volatility models

8.4.2 Materializing the spot/volatility cross-gamma P&L

The payoff proposed in this section, $\ln^2(S/S_0)$, once delta-hedged with S and vega-hegded using log contracts (or VSs) of the payoff's maturity materializes the realized covariance of $\ln S_t$ and $\widehat{\sigma}_T^2(t)$, weighted by $(T-t)$. The total P&L however also includes the realized variance of $\widehat{\sigma}_T^2(t)$, since the second derivative $\frac{d^2 P_{\text{BS}}}{d(\sigma^2)^2}$ does not vanish, even though the latter P&L is of order 2 in volatility of volatility.

An alternative measure of spot/volatility covariance

There does exist, however, a payoff that exactly materializes the realized covariance of S_t – not $\ln S_t$ – and $\widehat{\sigma}_T^2(t)$, weighted by $(T-t)$ – with no other spurious P&Ls.

It is the payoff $S\ln S$, which we examine in Chapter 3, Section 3.1.9, when we discuss the vega-hedge of FVAs, that is of cliquets whose vega is proportional to S_t.

The price R^T of a $S\ln S$ contract of maturity T is given by equation 3.19, page 117:

$$R^T\,(t,S) \;=\; Se^{-q(T-t)}\left(\ln S + (r-q)\,(T-t) + \frac{(T-t)\sigma^2}{2}\right) \tag{8.1}$$

The mixed derivative $\frac{d^2R}{dSd(\sigma^2)}$ is given by:

$$\frac{d^2R}{dSd(\sigma^2)} \;=\; \frac{1}{2}e^{-q(T-t)}(T-t) \tag{8.2}$$

and we also have:

$$\frac{d^2R}{d(\sigma^2)^2} \;=\; 0$$

Consider a long position in two units of the $S\ln S$ contract, *risk-managed at the log-contract implied volatility* $\widehat{\sigma}_T\,(t)$, dynamically delta-hegded and vega-hedged with log-contracts.

There are no delta and vega P&Ls.

We are vega-hedged and are risk-managing two European payoffs of the same maturity ($S\ln S$ and log contract) as the same implied volatility: cancellation of vega implies cancellation of gamma, and of theta as well.

The only P&L left is thus the $S_t/\widehat{\sigma}_T^2(t)$ cross-gamma P&L. Using (8.2), our final P&L, suitably compounded at T, reads:

$$\sum e^{r(T-t_i)}e^{-q(T-t_i)}(T-t_i)\,\delta S_i\,\delta(\widehat{\sigma}_T^2(t_i))$$

which can be rewritten as:

$$\sum (T-t_i)\,\delta F_i\,\delta(\widehat{\sigma}_T^2(t_i)) \tag{8.3}$$

where $F_i = S_i e^{(r-q)(T-t_i)}$ is the forward at time t_i for maturity T.

This result was first obtained by A. Neuberger; see [1] and [2].

Market price of the realized spot/volatility covariance

How much should we charge a client for (the exotic) payoff (8.3)? If the implied volatility of the $S\ln S$ contract were equal to that of the log contract, this payoff would be free. The price P we should charge is thus the market price of the $S\ln S$ contract minus its value calculated using the log-contract implied volatility maturity $\widehat{\sigma}_T$.

Using expression (8.1) of the price of the $S\ln S$ contract as a function of volatility, we get the market price of the realized spot/volatility covariance in (8.3):

$$\begin{aligned} P &= Se^{-qT}\left(\widehat{\sigma}_{S\ln S}^2 - \widehat{\sigma}_{\ln S}^2\right)T \\ &= e^{-rT}F^T\left(\widehat{\sigma}_{S\ln S}^2 - \widehat{\sigma}_{\ln S}^2\right)T \end{aligned}$$

where F^T is the forward for maturity T.

The implied volatilities of the log contract and $S\ln S$ contract are given by formulas (4.21), page 142 and (4.22), page 143, as a function of vanilla implied volatilities:

$$\begin{aligned} \widehat{\sigma}_{\ln S}^2 &= \int_{-\infty}^{+\infty} dy\,\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\widehat{\sigma}_{K(y)T}^2 \\ y(K) &= \frac{\ln\left(\frac{F_T}{K}\right)}{\widehat{\sigma}_{KT}\sqrt{T}} - \frac{\widehat{\sigma}_{KT}\sqrt{T}}{2} \end{aligned}$$

and:

$$\begin{aligned} \widehat{\sigma}_{S\ln S}^2 &= \int_{-\infty}^{+\infty} dy\,\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\widehat{\sigma}_{K(y)T}^2 \\ y(K) &= \frac{\ln\left(\frac{F_T}{K}\right)}{\widehat{\sigma}_{KT}\sqrt{T}} + \frac{\widehat{\sigma}_{KT}\sqrt{T}}{2} \end{aligned}$$

Typically, for equity smiles, $\widehat{\sigma}_{S\ln S} < \widehat{\sigma}_{\ln S}$, thus $P < 0$: the implied level of spot/volatility covariance is negative.

Bibliography

[1] Neuberger, A.: *The slope of the smile, and the comovement of volatility and returns*, available at SSRN: http://ssrn.com/abstract=1358863, 2009.

[2] Fukasawa, M.: *Volatility Derivatives and Model-free Implied Leverage*, International Journal of Theoretical and Applied Finance, **17**(1) 1450002, 2014.