Static / dynamic properties of stochastic volatility models:

a structural connection

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Outline

- Stochastic volatility models produce a smile *AND* generate a dynamics for implied volatilities

- Is there a connection of a structural nature ?
- Enter the *Skew Stickiness Ratio*

    - Long-maturity smiles
        - an expression linking the decay of the ATM and the SSR
    - Historical dynamics of market smiles ?

    - Short-maturity smiles
        - The *realized* skew
        - Materializing the difference between implied and realized skew as a P&L

- Conclusion

- General expression of a stochastic volatility model – no local vol component:

$$dS_t^\omega = (r-q)S_t^\omega dt + \sqrt{\xi_t^t}\, S_t^\omega dZ_t$$

$$d\xi_t^T = \omega\, \xi_t^T \sum_k \lambda_{kt}^T(\xi_t)\, dW_t^k$$

- At order 1 in ω:

$$\xi_t^T = \xi_0^T \left(1 + \omega \int_0^T \sum_k (\lambda_{k\tau}^T)_0\, dW_\tau^k\right)$$

  – In particular, for the instantaneous variance:

$$\xi_t^t = \xi_0^t + \delta\xi_t$$

$$\delta\xi_t = \omega\, \xi_0^t \int_0^t \sum_k (\lambda_{k\tau}^t)_0\, dW_\tau^k$$

  – ATMF skew as a function of skewness s_T of $x_T = \ln\left(\frac{S_T}{F_T}\right)$: $\mathcal{S}_T = \left.\frac{d\hat{\sigma}_{KT}}{d\ln K}\right|_F \approx \frac{s_T}{6\sqrt{T}}$

  with $s_T = \frac{M_3^T}{(M_2^T)^{3/2}}$, $M_n^T = (x_T - <x_T>)^n$

  – At order 1 in ω: $x_T - <x_T> = \int_0^T \sqrt{\xi_0^t}\, dZ_t + \frac{1}{2}\int_0^T \frac{\delta\xi_t}{\sqrt{\xi_0^t}} dZ_t - \frac{1}{2}\int_0^T \delta\xi_t dt$

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The ATMF skew - 2

$$M_2^T = \int_0^T \sqrt{\xi_0^t}\, dt$$

$$M_3^T = \frac{3}{2} E\left[\left(\int_0^T \sqrt{\xi_0^t}\, dZ_t\right)^2 \left(-\int_0^T \delta\xi_t dt + \int_0^T \frac{\delta\xi_t}{\sqrt{\xi_0^t}} dZ_t\right)\right]$$

- Computing the expectation:

$$M_T^3 = 3\omega \int_0^T dt\, \xi_0^t \int_0^t \sqrt{\xi_0^\tau} \sum_k \rho_{iS} (\lambda_{i\tau}^t)_0\, d\tau = 3 \int_0^T dt \left(\int_0^t E\left[\frac{dS_\tau^0}{S_\tau^0} \delta\xi_t\right]\right)$$

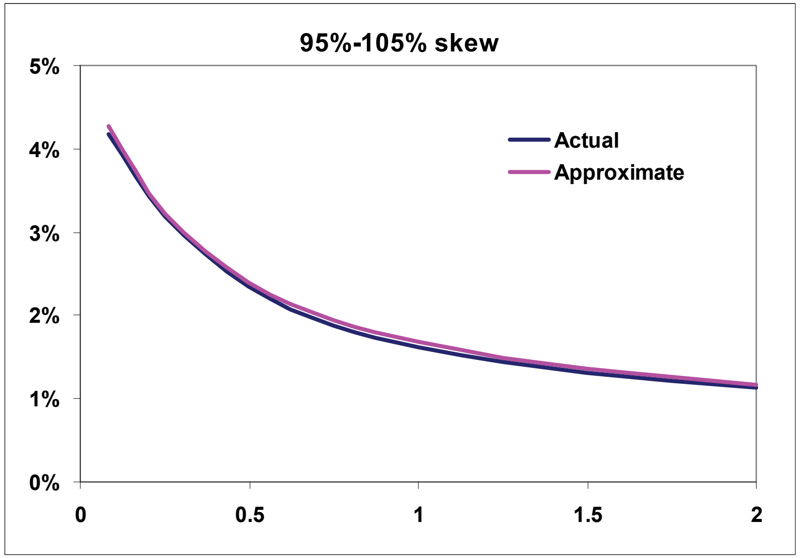
  – intuition: $M_T^3 = \left\langle (\Sigma r_i)^3 \right\rangle = \Sigma < r_i r_j r_k > = 3 \sum_{j>i} < r_i r_j^2 >$

- Introduce spot/volatility covariance function:

$$f(\tau,t) = \frac{1}{d\tau} E\left[\frac{dS_\tau^0}{S_\tau^0} \delta\xi_t\right]$$

- At order 1 in volatility-of-volatility, ATMF skew is given by:

$$S_T = \frac{1}{2\sqrt{T}} \frac{\int_0^T dt \int_0^t f(\tau,t)\, d\tau}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}}$$



The SSR - 1

- How much does the ATMF volatility move when the spot moves – in units of the skew?

- Market-makers use following ratio: $r_T = \frac{1}{\left.\frac{d\hat{\sigma}_{KT}}{d\ln K}\right|_F} \frac{d\hat{\sigma}_{FT}}{d\ln S}$
  - $r_T = 1$: " sticky-strike " regime
  - $r_T = 0$: " sticky-delta " regime

- Introduce Skew Stickiness Ratio: $$R_T = \frac{1}{\left.\frac{d\hat{\sigma}_{KT}}{d\ln K}\right|_F} \frac{E\left[d\hat{\sigma}_{FT}\; d\ln S\right]}{E[(d\ln S)^2]}$$

  - Jump / Lévy models : $R_T = 0$
  - Local vol, weak skew : $R_T = 2$
  - Stochastic volatility, short maturity & weak skew: $R_T = 2$

The SSR - 2

- At order 1 in volatility-of-volatility

  – use VS volatility instead of ATMF volatility:

$$E[d\ln S_\tau \, d\hat{\sigma}_T] = \frac{1}{2\hat{\sigma}_T T}\int_0^T E[d\ln S \, d\xi^t]dt = \frac{1}{2\hat{\sigma}_T T}\int_0^T E[\frac{dS_0^0}{S_0^0}\,\delta\xi_t]dt = \frac{1}{2\hat{\sigma}_T T}\int_0^T f(0,t)\,dt$$

  – Final expression for the SSR, at order 1 in volatility-of-volatility:

$$R_T = \frac{\int_0^T \xi_0^t\,dt}{\xi_0^0 T}\,\frac{T\int_0^T f(0,t)\,dt}{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}$$

$$S_T = \frac{1}{2\sqrt{T}}\frac{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}}$$

- Take short-maturity limit:

$$R_0 = \lim_{T\to 0}\frac{T\int_0^T f(0,t)\,dt}{\int_0^T dt\int_0^t f(\tau,t)\,d\tau} = 2$$

$$S_0 = \lim_{T\to 0}\frac{1}{2\sqrt{T}}\frac{\int_0^T dt\int_0^t f(\tau,t)\,d\tau}{\left(\int_0^T \xi_0^t dt\right)^{\frac{3}{2}}} = \frac{f(0,0)}{4(\xi_0^0)^{3/2}}$$

  – Skew tends to a finite value which measures the covariance function at origin

  – SSR tends to universal value: 2

Bounds for the SSR

- Let us make some additional assumptions:
  - VS curve is flat: $\xi_0^T \equiv \xi_0$
  - Stochastic volatility model is time-homogeneous: $f(\tau,t) \equiv f(t-\tau)$
  - ATMF skew and SSR take simpler forms:

$$S_T = \frac{\int_0^T (T-t) f(t)}{2\xi_0^{3/2} T^2} \qquad R_T = \frac{\int_0^T f(t)\,dt}{\int_0^T (1-\frac{t}{T}) f(t)\,dt}$$

- Assume that $|f(t)|$ decreases monotonically towards zero. Rewrite R_T as $R_T = \frac{g(T)}{\frac{1}{T}\int_0^T g(t)\,dt}$, $g(t) = \int_0^t f(\tau)\,d\tau$
  - $\frac{g(t)}{g(T)} \leq 1$ implies $R_T \geq 1$
  - $\frac{g(t)}{g(T)} \geq \frac{t}{T}$ implies $R_T \leq 2$

⇨ Model-independent range for R_T : $R_T \in [1, 2]$

Scaling of the SSR

- Assume that for large t $f(t) \propto \frac{1}{t^{\gamma}}$
- Then, for large *T* two types of behaviour, depending on value of γ
  - Type I : If $\gamma > 1$

$$S_T \propto \frac{1}{T} \quad \textit{and} \quad \lim_{T\to\infty} R_T = 1$$

  - Type II : If $\gamma < 1$

$$S_T \propto \frac{1}{T^{\gamma}} \quad \textit{and} \quad \lim_{T\to\infty} R_T = 2-\gamma$$

  - exponential decay: Type I.

- Type I: scaling of S_T analogous to models with independent increments, however model becomes sticky-strike
- Both types of scaling can be summarized by:

$$S_T \propto \frac{1}{T^{2-R_\infty}}$$

where $R_\infty = \lim_{T\to\infty} R_T$

Type II scaling – in a model ?

- Consider a model of the following type (Bergomi, 2005): $d\xi_t^T = \omega \xi_t^T \sum w_i e^{-k_i(T-t)} dW_t^i$

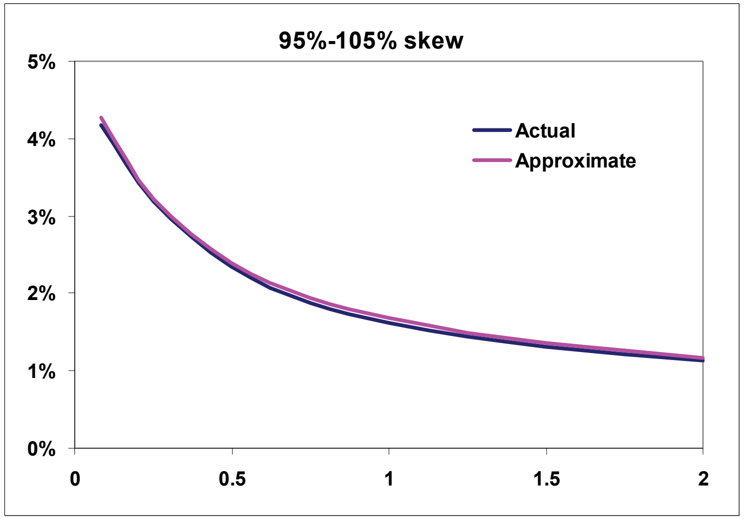
$$f(\tau) = \omega \xi_0^{3/2} \sum w_i \rho_{Si} e^{-k_i \tau}$$

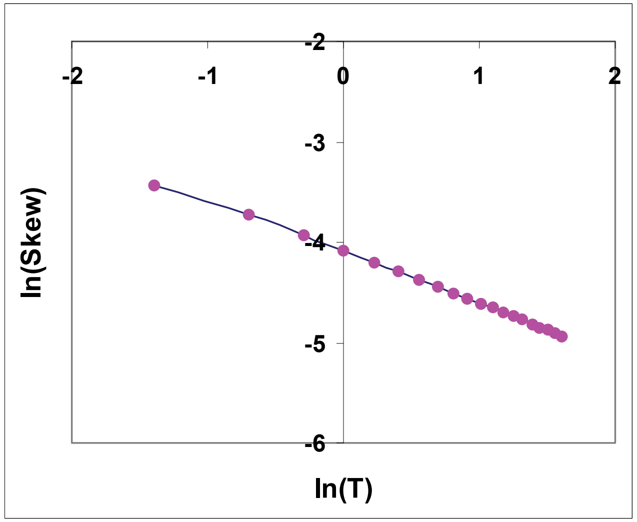
  – expressions for skew and SSR:

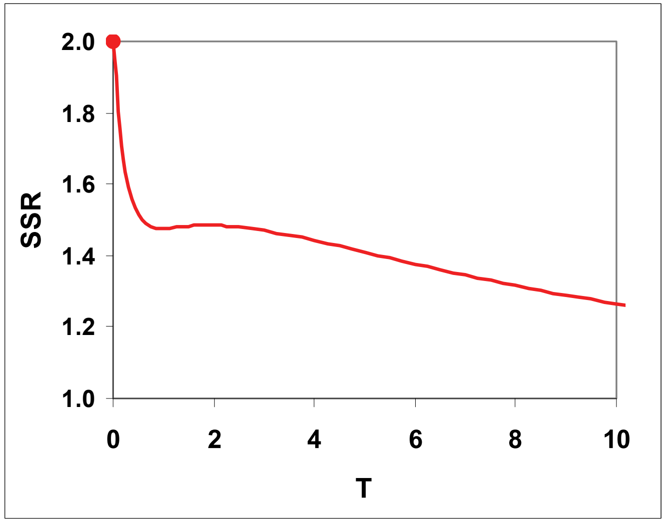
$$S_T = \frac{\omega}{2} \sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2} \qquad R_T = \frac{\sum w_i \rho_{Si} \frac{1 - e^{-k_i T}}{k_i T}}{\sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2}}$$

  – for large τ $f(\tau) \propto e^{-(\min_i k_i)\tau}$ ⇨ for (really) large T $S_T \propto \frac{1}{T}$, $R_T \to 1$

- By suitably choosing parameter values, get non-trivial scaling over wide range of maturities:
- Example : for intermediate range of maturities $S_T \propto 1/\sqrt{T}$ and $R_T \approx 1.5$



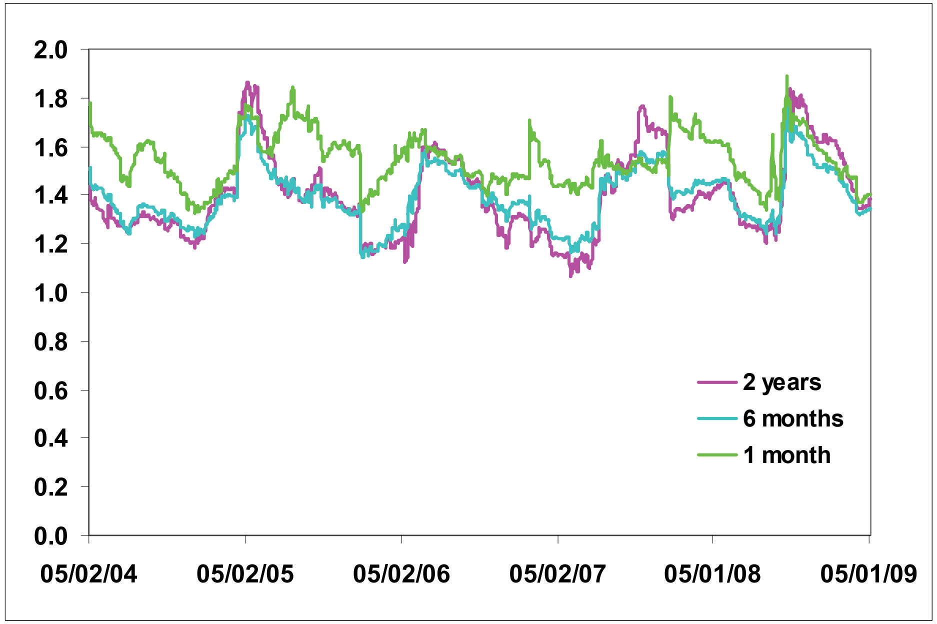




$k_1 = 8,\ k_2 = 0.35,\ w_1 = 72\%,\ w_2 = 28\%,\ \rho_{S1} = -70\%,\ \rho_{S2} = -35.7\%,\ w = 3.36$

Type II scaling – in the market ?

- Equity index smiles usually exhibit an ATMF skew that typically decays like $\frac{1}{\sqrt{T}}$
  - looks like Type II, but what about the SSR ?
  - For the Eurostoxx, 3-month running estimates of the SSR for maturities 1 month, 6 months, 2 years:



$$R_T = \frac{\sum (\hat{\sigma}_{i+1} - \hat{\sigma}_i) \ln\left(\frac{S_{i+1}}{S_i}\right)}{\sum \left.\frac{d\hat{\sigma}^i_{KT}}{d\ln S}\right|_F \ln\left(\frac{S_{i+1}}{S_i}\right)^2}$$

⇨ SSR usually in interval [1, 2]

⇨ For longer maturities, average value of the SSR ~ 1.5 – compatible with a decay of the skew ~ $\frac{1}{\sqrt{T}}$

⇨ Equity volatility markets seem to be of Type II

- One puzzle left: short-maturity SSR always lower than 2 – can this be arbitraged ?
  – is it possible materialize $2 - R_0$ as a P&L ? (or are we just plain wrong?)

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Arbing the realized SSR – 1

- SSR involves covariance of spot return & ATM vol return, and skew
  - ➩ Need model
    - – in which at least spot and ATM vol are dynamic
    - – that can consistently manage all strikes

- In Black Scholes model, P&L reads $P\&L = \frac{1}{2}S^2\frac{d^2P_K}{dS^2}\left(\left(\frac{\delta S}{S}\right)^2 - \sigma_K^2\,\delta t\right)$
  - – If we sell the 95, buy the 105
    - – zero Gamma
    - – " free " Theta
    - – how " free " is the Theta ? No way of assessing what the " fair " level of skew is

- We would like our P&L to read :

$$P\&L = \frac{1}{2}S^2\frac{d^2Q}{dS^2}\left(\left(\frac{\delta S}{S}\right)^2 - \sigma_S^2\,\delta t\right) + \frac{1}{2}\sigma_0^2\frac{d^2Q}{d\sigma_0^2}\left(\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 - \nu^2\,\delta t\right) + S\sigma_0\frac{d^2Q}{dSd\sigma_0}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0} - \rho\sigma_S\nu\,\delta t\right)$$

where σ_0 is the ATM vol and the break-even levels σ_S, ν, ρ do not depend on strike (unlike in the BS model).

- Parameterize smile for short, near-the-money options as:

$$\hat{\sigma}(x) = \sigma_0\left(1 + \alpha(\sigma_0)x + \frac{\beta(\sigma_0)}{2}x^2\right) \quad , \quad x = \ln(K/S)$$

Arbing the realized SSR – 2

- For a delta-hedged, σ_0 – hedged option, P&L reads:

$$P\&L = \frac{dQ}{dt}\delta t + \frac{1}{2}S^2\frac{d^2Q}{dS^2}\left(\frac{\delta S}{S}\right)^2 + \frac{1}{2}\sigma_0^2\frac{d^2Q}{d\sigma_0^2}\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 + S\sigma_0\frac{d^2Q}{dSd\sigma_0}\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}$$

- Our parameterization does not explicitly involve time
  - Our Theta is the same as the BS Theta, only distributed differently :

$$-\frac{dQ}{dt}\delta t = \frac{1}{2}S^2\frac{d^2P_K}{dS^2}\hat{\sigma}_K^2$$

$$= \frac{1}{2}S^2\frac{d^2Q}{dS^2}\sigma_S^2 + \frac{1}{2}\sigma_0^2\frac{d^2Q}{d\sigma_0^2}\nu^2 + S\sigma_0\frac{d^2Q}{dSd\sigma_0}\rho\sigma_S\nu$$

  - $\frac{dP_{BS}}{dt}, \frac{d^2P_{BS}}{dS^2}, \frac{d^2P_{BS}}{d\sigma^2}, \frac{d^2P_{BS}}{dSd\sigma}$ all start with same prefactor $\frac{SN'(d)}{\sigma\sqrt{T}}$ with $d = \frac{-x+\frac{\sigma^2T}{2}}{\sigma\sqrt{T}}$

  - Pull prefactor out and expand at order 2 in x and order 0 in T :

$$\frac{dQ}{dt} = -\aleph\cdot\sigma_0^2\left(1+\alpha x+\frac{\beta x^2}{2}\right)$$

$$\frac{1}{2}S^2\frac{d^2Q}{dS^2} = \aleph\cdot\left(1-3\alpha x+(6\alpha^2-\frac{5}{2}\beta)x^2\right)$$

$$\frac{1}{2}\sigma_0^2\frac{d^2Q}{d\sigma_0^2} = \aleph\cdot x^2$$

$$S\sigma_0\frac{d^2Q}{dSd\sigma_0} = 2\aleph\cdot\left(x-(2\alpha-\alpha')x^2\right)$$

where $\aleph = \frac{SN'(d)}{2\sigma_0\sqrt{T}}, \quad \alpha' = \frac{d\alpha}{d\ln\sigma_0}$

Arbing the realized SSR – 3

- Expression of P&L :

$$P\&L = \frac{dQ}{dt}\delta t + \frac{1}{2}S^2\frac{d^2Q}{dS^2}\left(\frac{\delta S}{S}\right)^2 + \frac{1}{2}\sigma_0^2\frac{d^2Q}{d\sigma_0^2}\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 + S\sigma_0\frac{d^2Q}{dSd\sigma_0}\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}$$

$$= \aleph\cdot\left[-\sigma_0^2\delta t\left(1+\alpha x+\frac{\beta x^2}{2}\right) + \left(1-3\alpha x+(6\alpha^2-\frac{5}{2}\beta)x^2\right)\left(\frac{\delta S}{S}\right)^2 + x^2\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2 + 2\left(x-(2\alpha-\alpha')x^2\right)\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right]$$

- now, find the break-even level that make P&L vanish $\forall x$: $\left\langle\frac{\delta S^2}{S^2}\right\rangle \equiv \sigma_S^2\delta t,\ \left\langle\frac{\delta\sigma_0^2}{\sigma_0^2}\right\rangle \equiv \nu^2\delta t,\ \left\langle\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right\rangle = \rho\sigma_S\nu$

$$P\&L = \aleph\cdot\left[-\sigma_0^2\left(1+\alpha x+\frac{\beta x^2}{2}\right) + \left(1-3\alpha x+(6\alpha^2-\frac{5}{2}\beta)x^2\right)\sigma_S^2 + x^2\nu^2 + 2\left(x-(2\alpha-\alpha')x^2\right)\rho\sigma_S\nu\right]\delta t$$

$$= \aleph\cdot\left[(-\sigma_0^2+\sigma_S^2) + (-\sigma_0^2\alpha-3\alpha\sigma_S^2+2\rho\sigma_S\nu)x + \left(-\sigma_0^2\frac{\beta}{2}+(6\alpha^2-\frac{5}{2}\beta)\sigma_S^2+\nu^2-2(2\alpha-\alpha')\rho\sigma_S\nu\right)x^2\right]\delta t$$

- Get following conditions:

$$\sigma_S = \sigma_0$$
$$\rho\nu = 2\alpha\sigma_0$$
$$\nu^2 = \sigma_0^2(3\beta+2\alpha^2-4\alpha\alpha')$$

⇨ break-even level of spot vol is ATM vol

⇨ break-even level of spot / ATM vol covariance is given by ATM skew

⇨ relationship between vol-of-vol and smile curvature is model-dependent

Arbing the realized SSR – 4

$$\begin{aligned}\sigma_S &= \sigma_0 \\ \rho\nu &= 2\alpha\sigma_0 \\ \nu^2 &= \sigma_0^2(3\beta + 2\alpha^2 - 4\alpha\alpha')\end{aligned}$$

- Examples of admissible models :

- Lognormal dynamics for σ_0
  - ρ, ν constant implies that $\alpha(\sigma_0) = \frac{a}{\sigma_0}, \quad \beta(\sigma_0) = \frac{b}{\sigma_0^2}$

$$\begin{aligned}\hat{\sigma}(x) &= \sigma_0\left(1 + \frac{a}{\sigma_0}x + \frac{b}{2\sigma_0^2}x^2\right) \\ \rho\nu &= 2a \\ \nu &= \sqrt{3b + 6a^2}\end{aligned}$$

➪ skew does not depend on vol

➪ curvature is inversely proportional to vol

- Normal dynamics for σ_0
  - $\nu \propto \frac{1}{\sigma_0}$ implies that $\alpha(\sigma_0) = \frac{a}{\sigma_0^2}, \quad \beta(\sigma_0) = \frac{b}{\sigma_0^4}$

$$\begin{aligned}\hat{\sigma}(x) &= \sigma_0\left(1 + \frac{a}{\sigma_0^2}x + \frac{b}{2\sigma_0^4}x^2\right) \\ \rho\mu &= 2a \\ \mu &= \sqrt{3b + 10a^2}\end{aligned}$$

➪ skew inversely proportional to vol

➪ curvature is inversely proportional to vol^3

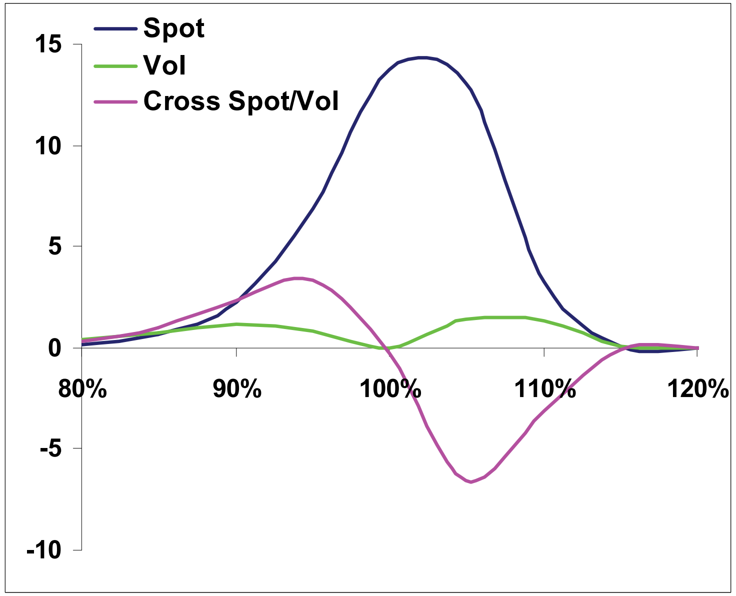
Arbing the realized SSR – 5

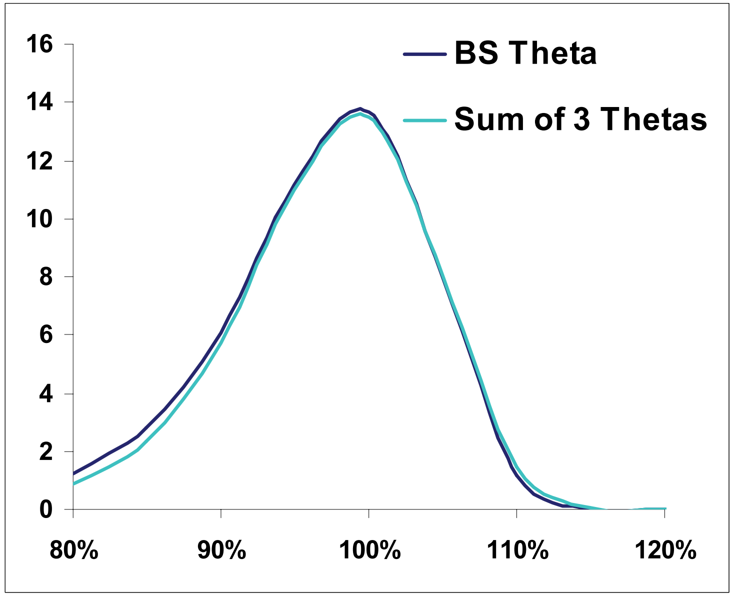
- Use lognormal model for σ_0 :
  - P&L accounting:

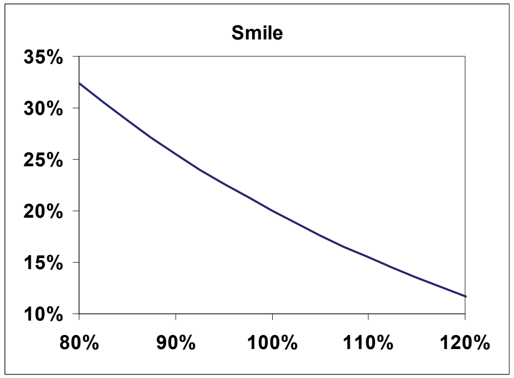
$$P\&L = \frac{1}{2}S^2\frac{d^2Q}{dS^2}\left(\left(\frac{\delta S}{S}\right)^2-\sigma_0^2\,\delta t\right) + \frac{1}{2}\sigma_0^2\frac{d^2Q}{d\sigma_0^2}\left(\left(\frac{\delta\sigma_0}{\sigma_0}\right)^2-(3b+6a^2)\,\delta t\right) + S\sigma_0\frac{d^2Q}{dSd\sigma_0}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}-2a\sigma_0\,\delta t\right)$$

$$\underbrace{S\sigma_0\frac{d^2Q}{dSd\sigma_0}\left(\frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}-2\sigma_0\left.\frac{d\hat{\sigma}}{d\ln K}\right|_S\delta t\right)}_{\textit{Arb P\&L}}$$

- Does it work ? How do the 3 Thetas add up to the BS Theta ?
  - Take 1 month option, $\sigma_0=20\%,\ a=-10\%,\ b=0.4\%$
  - Look at 3 Thetas + sum







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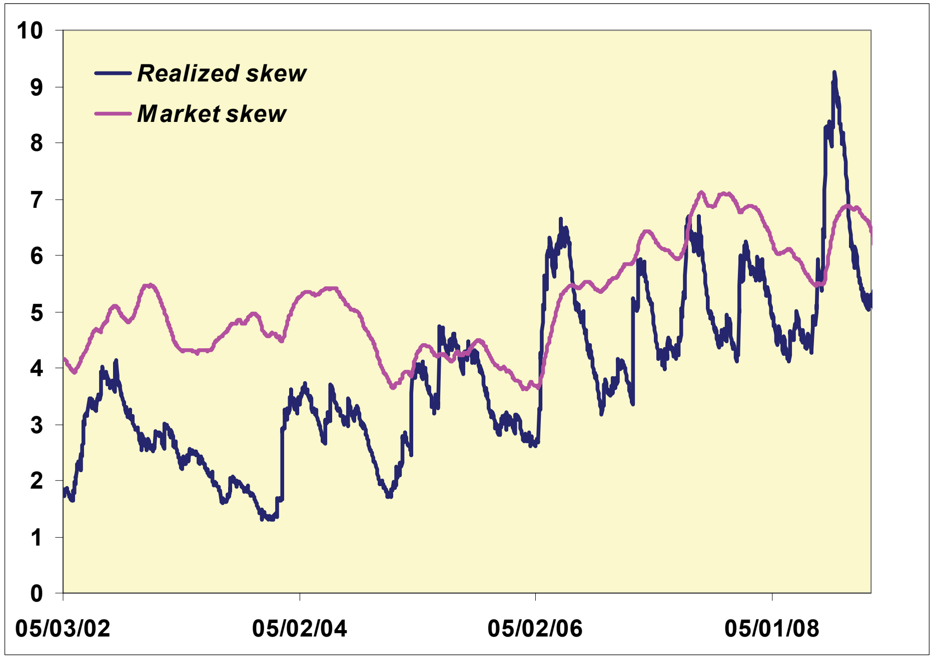
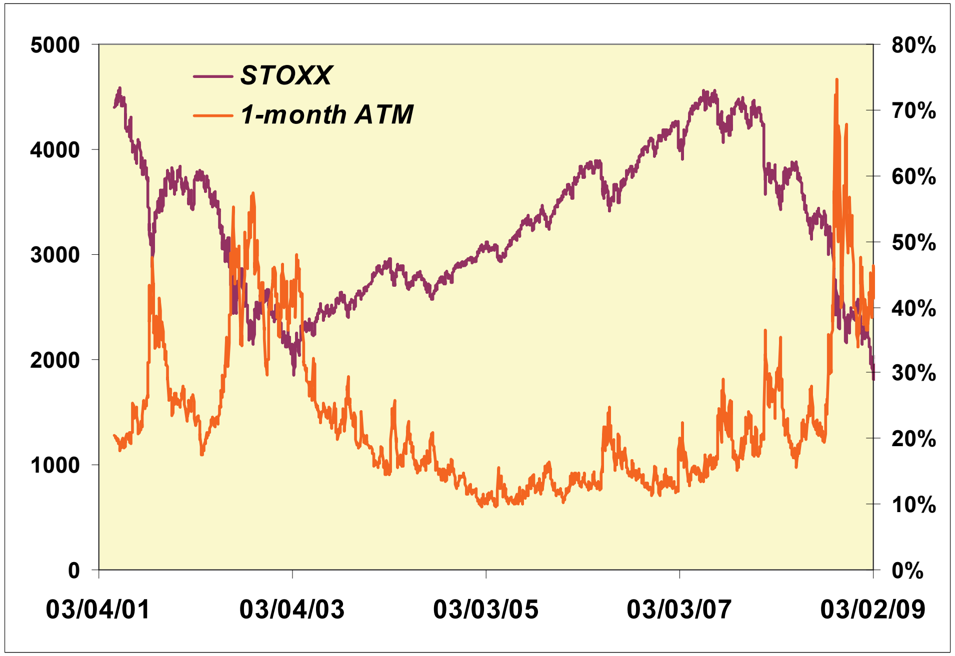
Arbing the realized SSR – 6

- Try out arbitrage on Stoxx50
  - introduce " realized " skew:

$$\left.\frac{d\hat{\sigma}}{d\ln K}\right|_S^{Realized} = \frac{1}{2\sigma_0\delta t}\left\langle \frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right\rangle$$

  - implied versus " realized " 1-month 95 / 105 skew





  - Back-test strategy:
    - every day short one 95 option, buy 105 option so that Spot Gamma = zero (~ 0.6)
    - carry position until next day, then unwind & restart

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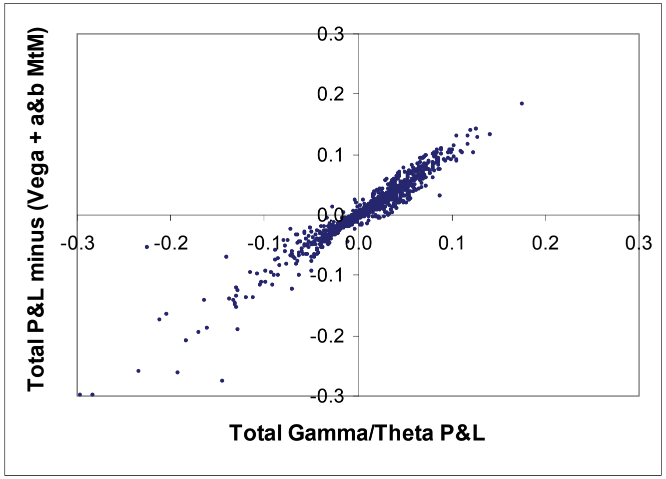
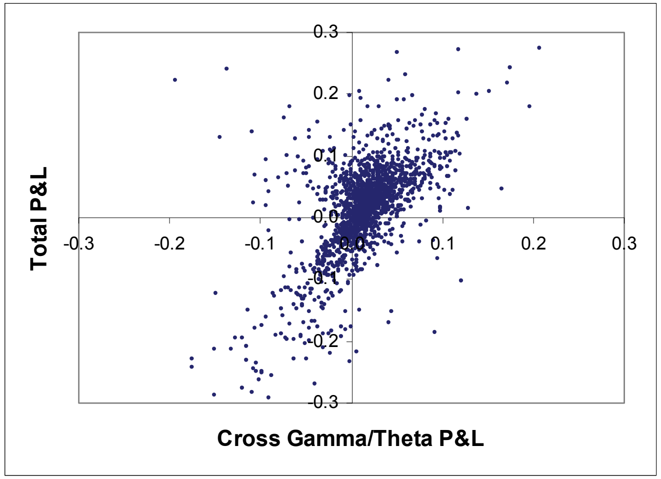
Arbing the realized SSR – 7

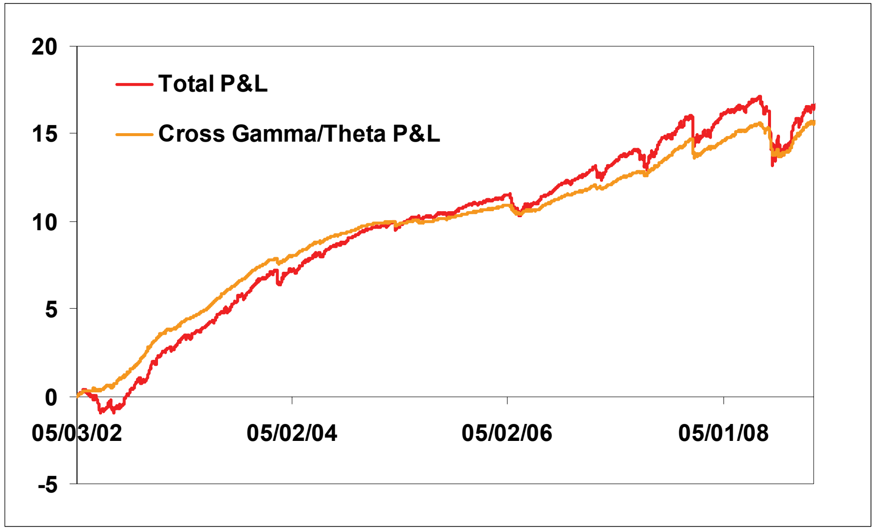
- P&L analysis : in addition to cross-Gamma P&L:
  - Vega P&L
  - vol Gamma / Theta P&L
  - P&L generated by remarking to market a and b ⇨ tells us whether our model is misspecified

- Scatter plots of daily P&Ls:





- Cumulative P&L:



Conclusion

- In stochastic volatility models, at order 1 in the vol-of-vol, the Skew Stickiness Ratio and the rate of decay of the ATMF skew are related through the spot/vol covariance function

- For time-homogeneous model + flat VS term-structure:
  - SSR is bounded : $\mathcal{R}_T \in [1, 2]$
  - 2 types of models
    - Type I $\quad \mathcal{S}_T \propto \frac{1}{T}$ *and* $\lim_{T\to\infty} \mathcal{R}_T = 1$
    - Type II $\quad \mathcal{S}_T \propto \frac{1}{T^\gamma}$ *and* $\lim_{T\to\infty} \mathcal{R}_T = 2-\gamma$

$$\mathcal{S}_T \propto \frac{1}{T^{2-\mathcal{R}_\infty}}$$

- Index volatility markets' behaviour consistent with Type II

- For short maturity smiles, SSR = 2
  - Markets display a realized SSR < 2
  - Introduce the notion of " realized skew "

    realized covariance of spot & implied vol
  - $2-\mathcal{R}_0$ mismatch can be materialized as a P&L

$$\left.\frac{d\hat{\sigma}}{d\ln K}\right|_S^{Realized} = \frac{1}{2\sigma_0\delta t}\left\langle \frac{\delta S}{S}\frac{\delta\sigma_0}{\sigma_0}\right\rangle$$