Static / dynamic properties of stochastic volatility models:

a structural connection

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Outline

• Stochastic volatility models produce a smile AND generate a dynamics for implied volatilities

• Is there a connection of a structural nature?
  • Enter the Skew Stickiness Ratio

  • Long-maturity smiles
    • an expression linking the decay of the ATM and the SSR
    • Historical dynamics of market smiles?

  • Short-maturity smiles
    • The realized skew
    • Materializing the difference between implied and realized skew as a P&L

• Conclusion
The ATMF skew - 1

- General expression of a stochastic volatility model – no local vol component:

\[
    dS_t^\omega = (r - q)S_t^\omega dt + \sqrt{\xi_t} S_t^\omega dZ_t
\]

\[
    d\xi_t^T = \omega \xi_t^T \sum_k \lambda_{kt}^T (\xi_t) dW^k_t
\]

- At order 1 in \( \omega \):

\[
    \xi_t^T = \xi_0^T \left( 1 + \omega \int_0^T \sum_k (\lambda_{kt})_0 dW^k_t \right)
\]

- In particular, for the instantaneous variance:

\[
    \xi_t = \xi_0 + \delta \xi_t
\]

\[
    \delta \xi_t = \omega \xi_0 \int_0^T \sum_k (\lambda_{kt})_0 dW^k_t
\]

- ATMF skew as a function of skewness \( s_T \) of \( x_T = \ln \left( \frac{S_T}{F_T} \right) \):

\[
    S_T = \frac{d \hat{\sigma}_{KT}}{d \ln K} \bigg|_F \approx \frac{s_T}{6 \sqrt{T}}
\]

with \( s_T = \frac{M_3^T}{(M_2^T)^{3/2}} \), \( M_n^T = (x_T - < x_T >)^n \)

- At order 1 in \( \omega \):

\[
    x_T - < x_T > = \int_0^T \sqrt{\xi_0} dZ_t + \frac{1}{2} \int_0^T \frac{\delta \xi_t}{\sqrt{\xi_0}} dZ_t - \frac{1}{2} \int_0^T \delta \xi_t dt
\]
The ATMF skew - 2

\[ M_2^T = \int_0^T \sqrt{\xi(t)} dt \]

\[ M_3^T = \frac{3}{2} E \left[ \left( \int_0^T \sqrt{\xi(t)} dt \right)^2 \right. \left. \left( - \int_0^T \delta \xi(t) dt + \int_0^T \frac{\delta \xi(t) dZ_t}{\sqrt{\xi(t)}} \right) \right] \]

- Computing the expectation:

\[ M_3^T = 3 \int_0^T dt \xi(t) \int_0^T \xi(t) \sum_k \rho_{iS}(\lambda_{it})_0 d\tau = 3 \int_0^T dt \left( \int_0^t E \left[ \frac{dS_\tau^0}{S_\tau^0} \delta \xi(t) \right] \right) \]

- Intuition: \[ M_3^T = \langle (\Sigma_i)^3 \rangle = \Sigma < r_i r_j r_k > = 3 \sum_{j>i} < r_i r_j^2 > \]

- Introduce spot/volatility covariance function:

\[ f(\tau, t) = \frac{1}{d\tau} E \left[ \frac{dS_\tau^0}{S_\tau^0} \delta \xi(t) \right] \]

- At order 1 in volatility-of-volatility, ATMF skew is given by:

\[ S_T = \frac{1}{2^{\frac{3}{2}}} \int_0^T dt \left[ \int_0^t f(\tau, t) d\tau \right] \left( \int_0^T \xi(t) dt \right)^{-\frac{3}{2}} \]
• How much does the ATMF volatility move when the spot moves – in units of the skew?

• Market-makers use following ratio: 
  \[ r_T = \frac{1}{\frac{d\hat{\sigma}_{FT}}{d \ln K}_F} \frac{d\hat{\sigma}_{FT}}{d \ln S} \]
  
  – \( r_T = 1 \): "sticky-strike" regime
  – \( r_T = 0 \): "sticky-delta" regime

• Introduce Skew Stickiness Ratio:

\[ R_T = \frac{1}{\frac{d\hat{\sigma}_{FT}}{d \ln K}_F} \frac{E[d\hat{\sigma}_{FT} d\ln S]}{E[(d\ln S)^2]} \]

  – Jump / Lévy models: \( R_T = 0 \)
  – Local vol, weak skew: \( R_T = 2 \)
  – Stochastic volatility, short maturity & weak skew: \( R_T = 2 \)
The SSR - 2

• At order 1 in volatility-of-volatility

– use VS volatility instead of ATMF volatility:

\[
E[d\ln S_T \, d\hat{\sigma}_T] = \frac{1}{2\hat{\sigma}_T T} \int_0^T E[d\ln S \, d\xi^t] \, dt = \frac{1}{2\hat{\sigma}_T T} \int_0^T E\left[\frac{dS^0}{S^0} \, \delta\xi^t\right] \, dt = \frac{1}{2\hat{\sigma}_T T} \int_0^T f(0, t) \, dt
\]

– Final expression for the SSR, at order 1 in volatility-of-volatility:

\[
R_T = \frac{\int_0^T \xi^t_0 \, dt \int_0^T f(0,t) \, dt}{\int_0^T dt \int_0^t f(\tau,t) \, d\tau}
\]

\[
S_T = \frac{1}{2\sqrt{T}} \int_0^T dt \int_0^t f(\tau,t) \, d\tau \left( \int_0^T \xi^t_0 \, dt \right)^{3/2}
\]

• Take short-maturity limit:

\[
R_0 = \lim_{T \to 0} \frac{\int_0^T f(0,t) \, dt}{\int_0^T dt \int_0^t f(\tau,t) \, d\tau} = 2
\]

\[
S_0 = \lim_{T \to 0} \frac{1}{2\sqrt{T}} \int_0^T dt \int_0^t f(\tau,t) \, d\tau \left( \int_0^T \xi^t_0 \, dt \right)^{3/2} = \frac{f(0,0)}{4(\xi_0^0)^{3/2}}
\]

– Skew tends to a finite value which measures the covariance function at origin

– SSR tends to universal value: 2
Bounds for the SSR

• Let us make some additional assumptions:
  – VS curve is flat: \( \zeta_0^T \equiv \zeta_0 \)
  – Stochastic volatility model is time-homogeneous: \( f(\tau, t) \equiv f(t - \tau) \)
  – ATM skew and SSR take simpler forms:

\[
S_T = \frac{\int_0^T (T - t) f(t) \, dt}{2 \zeta_0^{3/2} T^2} \quad R_T = \frac{\int_0^T f(t) \, dt}{\int_0^T (1 - \frac{t}{T}) f(t) \, dt}
\]

• Assume that \(|f(t)|\) decreases monotonically towards zero. Rewrite \( R_T \) as \( R_T = \frac{g(T)}{\frac{1}{T} \int_0^T g(t) \, dt} \), \( g(t) = \int_0^T f(\tau) \, d\tau \)

\[
- \frac{g(t)}{g(T)} \leq 1 \text{ implies } R_T \geq 1
\]

\[
- \frac{g(t)}{g(T)} \geq \frac{t}{T} \text{ implies } R_T \leq 2
\]

\( \Rightarrow \) Model-independent range for \( R_T \): \( R_T \in [1, 2] \)
Scaling of the SSR

• Assume that for large $t$ \( f(t) \propto \frac{1}{t^\gamma} \)

• Then, for large $T$ two types of behaviour, depending on value of $\gamma$

  – Type I: If $\gamma > 1$
    \[
    S_T \propto \frac{1}{T} \quad \text{and} \quad \lim_{T \to \infty} R_T = 1
    \]

  – Type II: If $\gamma < 1$
    \[
    S_T \propto \frac{1}{T^\gamma} \quad \text{and} \quad \lim_{T \to \infty} R_T = 2 - \gamma
    \]

  – exponential decay: Type I.

• Type I: scaling of $S_T$ analogous to models with independent increments, however model becomes sticky-strike

• Both types of scaling can be summarized by:

\[
S_T \propto \frac{1}{T^{2-R_\infty}}
\]

where $R_\infty = \lim_{T \to \infty} R_T$
Consider a model of the following type (Bergomi, 2005):

\[ d\xi_t^T = \omega \xi_t^T \sum w_i e^{-k_i(T-t)} dW_i^t \]

\[ f(\tau) = \omega_{x_0}^{3/2} \sum w_i \rho Si e^{-k_i \tau} \]

- expressions for skew and SSR:

\[
S_T = \frac{\omega}{2} \sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2} \\
R_T = \frac{\sum w_i \rho_{Si} \frac{1 - e^{-k_i T}}{k_i T}}{\sum w_i \rho_{Si} \frac{k_i T - (1 - e^{-k_i T})}{(k_i T)^2}}
\]

- for large \( \tau \) \( f(\tau) \propto e^{-(\min_i k_i) \tau} \) \( \Rightarrow \) for (really) large \( T \) \( S_T \propto \frac{1}{T}, \ R_T \rightarrow 1 \)

- By suitably choosing parameter values, get non-trivial scaling over wide range of maturities:

- Example: for intermediate range of maturities \( S_T \propto 1/\sqrt{T} \) and \( R_T \approx 1.5 \)

\[ k_1 = 8, \ k_2 = 0.35, \ w_1 = 72\%, \ w_2 = 28\%, \ \rho_{S1} = -70\%, \ \rho_{S2} = -35.7\%, \ w = 3.36 \]
Type II scaling – in the market?

- Equity index smiles usually exhibit an ATMF skew that typically decays like $\frac{1}{\sqrt{T}}$.
  - looks like Type II, but what about the SSR?
  - For the Eurostoxx, 3-month running estimates of the SSR for maturities 1 month, 6 months, 2 years:

$$R_T = \sum \frac{d\hat{\sigma}_i}{d\ln S} \ln \left( \frac{S_{i+1}}{S_i} \right)^2$$

⇒ SSR usually in interval [1, 2]

⇒ For longer maturities, average value of the SSR $\sim 1.5$ – compatible with a decay of the skew $\sim \frac{1}{\sqrt{T}}$

⇒ Equity volatility markets seem to be of Type II

- One puzzle left: short-maturity SSR always lower than 2 – can this be arbitraged?
  - is it possible materialize $2 - R_0$ as a P&L? (or are we just plain wrong?)
Arbing the realized SSR – 1

- SSR involves covariance of spot return & ATM vol return, and skew
  - Need model
    - in which at least spot and ATM vol are dynamic
    - that can consistently manage all strikes

- In Black Scholes model, P&L reads
  \[ P & L = \frac{1}{2} S^2 \left( \frac{d^2 P_K}{dS^2} \left( \frac{\delta S}{S} \right)^2 - \sigma_K^2 \delta t \right) \]
  - If we sell the 95, buy the 105
    - zero Gamma
    - " free " Theta
    - how " free " is the Theta? No way of assessing what the " fair " level of skew is

- We would like our P&L to read:
  \[ P & L = \frac{1}{2} S^2 \left( \frac{d^2 Q}{dS^2} \left( \frac{\delta S}{S} \right)^2 - \sigma_S^2 \delta t \right) + \frac{1}{2} \sigma_0^2 \frac{d^2 Q}{d\sigma_0^2} \left( \frac{\delta \sigma_0}{\sigma_0} \right)^2 - \nu^2 \delta t \]  
  \[ + \sigma_0 \frac{d^2 Q}{dSd\sigma_0} \left( \frac{\delta S}{S} \frac{\delta \sigma_0}{\sigma_0} - \rho \sigma_S \nu \delta t \right) \]
  where \( \sigma_0 \) is the ATM vol and the break-even levels \( \sigma_S, \nu, \rho \) do not depend on strike (unlike in the BS model).

- Parameterize smile for short, near-the-money options as:
  \[ \hat{\sigma}(x) = \sigma_0 \left( 1 + a(\sigma_0) x + \frac{b(\sigma_0)}{2} x^2 \right), \quad x = \ln(K/S) \]
Arbing the realized SSR – 2

• For a delta-hedged, $\sigma_0$ – hedged option, P&L reads:

$$P & L = \frac{dQ}{dt} \delta_t + \frac{1}{2} S^2 \frac{d^2Q}{dS^2} \left( \frac{\delta S}{S} \right)^2 + \frac{1}{2} \sigma_0^2 \frac{d^2Q}{d\sigma_0^2} \left( \frac{\delta \sigma_0}{\sigma_0} \right)^2 + S \sigma_0 \frac{d^2Q}{dSd\sigma_0} \frac{\delta S}{S} \frac{\delta \sigma_0}{\sigma_0}$$

• Our parameterization does not explicitly involve time

  – Our Theta is the same as the BS Theta, only distributed differently:

$$-\frac{dQ}{dt} \delta_t = \frac{1}{2} S^2 \frac{d^2P}{dS^2} \sigma_K^2$$

$$= \frac{1}{2} S^2 \frac{d^2Q}{dS^2} \sigma_S^2 + \frac{1}{2} \sigma_0^2 \frac{d^2Q}{d\sigma_0^2} v^2 + S \sigma_0 \frac{d^2Q}{dSd\sigma_0} \rho \sigma_S v$$

$$- \frac{dP_{BS}}{dt}, \frac{d^2P_{BS}}{dS^2}, \frac{d^2P_{BS}}{d\sigma^2}, \frac{d^2P_{BS}}{dSd\sigma}$$ all start with same prefactor $\frac{S \sigma_0}{\sigma_0} \frac{(d)}{\sigma T}$ with $d = \frac{-x + \sigma^2 T}{\sigma T}$

– Pull prefactor out and expand at order 2 in $x$ and order 0 in $T$:

$$\frac{dQ}{dt} = -\mathcal{N} \cdot \sigma_0^2 \left( 1 + \alpha x + \beta x^2 \right)$$

$$\frac{1}{2} S^2 \frac{d^2Q}{dS^2} = \mathcal{N} \cdot \left( 1 - 3\alpha x + (6\alpha^2 - \frac{5}{2} \beta)x^2 \right)$$

$$\frac{1}{2} \sigma_0^2 \frac{d^2Q}{d\sigma_0^2} = \mathcal{N} \cdot x^2$$

$$S\sigma_0 \frac{d^2Q}{dSd\sigma_0} = 2\mathcal{N} \cdot \left( x - (2\alpha - \alpha')x^2 \right)$$

where $\mathcal{N} = \frac{S \sigma_0 (d)}{2\sigma_0 \sqrt{T}}$, $\alpha' = \frac{d\alpha}{d \ln \sigma_0}$
Arbing the realized SSR – 3

• Expression of P&L:

\[
P & L = \frac{dQ}{dt} \delta t + \frac{1}{2} \sigma_0^2 \frac{d^2 Q}{dS^2} \left( \frac{\delta S}{S} \right)^2 + \frac{1}{2} \frac{\delta \sigma_0}{\sigma_0} \frac{d^2 Q}{d\sigma_0^2} \left( \frac{\delta \sigma_0}{\sigma_0} \right)^2 + S \sigma_0 \frac{d^2 Q}{dSd\sigma_0} \frac{\delta S}{S} \frac{\delta \sigma_0}{\sigma_0}
\]

\[
= \mathbb{N} \cdot \left[ -\sigma_0^2 \delta t \left( 1 + \alpha x + \frac{\beta x^2}{2} \right) + \left( 1 - 3 \alpha x + (6 \alpha^2 - \frac{5}{2} \beta) x^2 \right) \frac{\delta S}{S} \right]^2 + x^2 \left( \frac{\delta \sigma_0}{\sigma_0} \right)^2 + 2 \left( x - (2 \alpha - \alpha') x^2 \right) \frac{\delta S}{S} \frac{\delta \sigma_0}{\sigma_0}
\]

• now, find the break-even level that make P&L vanish \( \forall x: \delta S^2 = \sigma_S^2 \delta t, \frac{\delta \sigma_0^2}{\sigma_0^2} = \nu^2 \delta t, \frac{\delta S}{S} \frac{\delta \sigma_0}{\sigma_0} = \rho \sigma_S \nu \)

\[
P & L = \mathbb{N} \cdot \left[ -\sigma_0^2 \left( 1 + \alpha x + \frac{\beta x^2}{2} \right) + \left( 1 - 3 \alpha x + (6 \alpha^2 - \frac{5}{2} \beta) x^2 \right) \sigma_S^2 + x^2 \nu^2 + 2 \left( x - (2 \alpha - \alpha') x^2 \right) \rho \sigma_S \nu \right] \delta t
\]

\[
= \mathbb{N} \cdot \left[ (-\sigma_0^2 + \sigma_S^2) + (-\sigma_0^2 \alpha - 3 \alpha \sigma_S^2 + 2 \rho \sigma_S \nu) x + \left( -\sigma_0^2 \frac{\beta}{2} + (6 \alpha^2 - \frac{5}{2} \beta) \sigma_S^2 + \nu^2 - 2 (2 \alpha - \alpha') \rho \sigma_S \nu \right) x^2 \right] \delta t
\]

• Get following conditions:

\[
\begin{align*}
\sigma_S &= \sigma_0 & \Rightarrow \text{break-even level of spot vol is ATM vol} \\
\rho \nu &= 2 \alpha \sigma_0 & \Rightarrow \text{break-even level of spot / ATM vol covariance is given by ATM skew} \\
\nu^2 &= \sigma_0^2 \left( 3 \beta + 2 \alpha^2 - 4 \alpha \alpha' \right)
\end{align*}
\]

\( \Rightarrow \text{relationship between vol-of-vol and smile curvature is model-dependent} \)
Examples of admissible models:

- **Lognormal dynamics for** $\sigma_0$
  
  $\rho, \nu$ constant implies that $a(\sigma_0) = \frac{a}{\sigma_0}$, $\beta(\sigma_0) = \frac{b}{\sigma_0^2}$

  $$\hat{\sigma}(x) = \sigma_0 \left(1 + \frac{a}{\sigma_0} x + \frac{b}{2\sigma_0^2} x^2\right) \Rightarrow \text{skew does not depend on vol}$$

  $$\rho \nu = 2a \Rightarrow \text{curvature is inversely proportional to vol}$$

  $$\nu = \sqrt{3b + 6a^2}$$

- **Normal dynamics for** $\sigma_0$

  $\nu \approx \frac{1}{\sigma_0}$ implies that $a(\sigma_0) = \frac{a}{\sigma_0^2}$, $\beta(\sigma_0) = \frac{b}{\sigma_0^4}$

  $$\hat{\sigma}(x) = \sigma_0 \left(1 + \frac{a}{\sigma_0^2} x + \frac{b}{2\sigma_0^4} x^2\right) \Rightarrow \text{skew inversely proportional to vol}$$

  $$\rho \mu = 2a \Rightarrow \text{curvature is inversely proportional to vol}^3$$

  $$\mu = \sqrt{3b + 10a^2}$$

\[
\begin{align*}
\sigma_S &= \sigma_0 \\
\rho \nu &= 2a \sigma_0 \\
v^2 &= \sigma_0^2 (3\beta + 2a^2 - 4aa')
\end{align*}
\]
Arbing the realized SSR – 5

• Use lognormal model for $\sigma_0$:
  
  P&L accounting:
  
  $$P & L = \frac{1}{2} S^2 \frac{d^2 Q}{dS^2} \left( \frac{\delta S}{S} \right)^2 - \sigma_0^2 \delta t + \frac{1}{2} \sigma_0^2 \frac{d^2 Q}{d\sigma_0^2} \left( \frac{\delta \sigma_0}{\sigma_0} \right)^2 - (3b + 6a^2) \delta t + S \sigma_0 \frac{d^2 Q}{dSd\sigma_0} \left( \frac{\delta S}{S} \delta \sigma_0 \sigma_0 - 2a \sigma_0 \delta \sigma \right)$$

• Does it work? How do the 3 Thetas add up to the BS Theta?
  
  – Take 1 month option, $\sigma_0 = 20\%$, $a = -10\%$, $b = 0.4\%$
  
  – Look at 3 Thetas + sum

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**Arb P&L**

**Spot**

**Vol**

**Cross Spot/Vol**

**BS Theta**

**Sum of 3 Thetas**

**Smile**

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Arbing the realized SSR – 6

- Try out arbitrage on Stoxx50
  - introduce "realized" skew:
    \[
    \frac{d\hat{\sigma}}{d \ln K}\bigg|_S^{\text{Realized}} = \left. \frac{1}{2\sigma_0 \delta t} \frac{\delta S}{S} \frac{\delta \sigma_0}{\sigma_0} \right. 
    \]
  - implied versus "realized" 1-month 95 / 105 skew

- Back-test strategy:
  - every day short one 95 option, buy 105 option so that Spot Gamma = zero (~ 0.6)
  - carry position until next day, then unwind & restart
Arbing the realized SSR – 7

- P&L analysis: in addition to cross-Gamma P&L:
  - Vega P&L
  - Vol Gamma / Theta P&L
  - P&L generated by remarking to market $a$ and $b \Rightarrow$ tells us whether our model is misspecified

- Scatter plots of daily P&Ls:

- Cumulative P&L:
Conclusions

• In stochastic volatility models, at order 1 in the vol-of-vol, the Skew Stickiness Ratio and the rate of decay of the ATMF skew are related through the spot/vol covariance function:

• For time-homogeneous model + flat VS term-structure:
  – SSR is bounded: \( R_T \in [1, 2] \)
  – 2 types of models
    – Type I: \( S_T \propto \frac{1}{T} \) and \( \lim_{T \to \infty} R_T = 1 \)
    – Type II: \( S_T \propto \frac{1}{T^\gamma} \) and \( \lim_{T \to \infty} R_T = 2 - \gamma \)

• Index volatility markets' behaviour consistent with Type II

• For short maturity smiles, SSR = 2
  – Markets display a realized SSR < 2
  – Introduce the notion of "realized skew"
    realized covariance of spot & implied vol
  – \( 2 - R_0 \) mismatch can be materialized as a P&L

\[
\text{Realized} \quad \frac{d\hat{\sigma}}{d \ln K}_S \bigg|_{S}^{\text{Realized}} = \frac{1}{2\sigma_0 \delta t} \begin{pmatrix} \delta S \\ \delta \sigma_0 \end{pmatrix} \]