Know your Theta – and your Gamma P&L

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Prologue

- LB meets exotic equity trader in Hong Kong:

  Trader  Could you spare 5 minutes so I show you my Theta over the last 2 weeks?
  LB    Sure, what’s wrong with your Theta?
  Trader  It’s messy.
  LB    Does look like white noise, indeed.

- LB back in Paris goes to other exotic equity trader:

  LB    Hi, in HK they showed me their Theta. It’s pretty messy. How come you’re not complaining about yours?
  Trader  Ours is messy too – is it supposed not to be? We average it over a few days.

- Question 1: how should we compute Theta?

- Question 0: which Theta should we compute?
Theta

- Θ calculated as \( P(t + \delta t, S) - P(t, S) \approx \frac{dP}{dt}\delta t \) is not a particularly useful number

- Look at P&L of delta-hedged position in model with only \( S \) as hedging instrument:

\[
P&L = \left( \frac{dP}{dt} - rP + (r - q)S \frac{dP}{dS} \right) \delta t + \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2
\]

- The job of a model is to pay us a deterministic amount during \( \delta \) to offset:
  - the financing costs of premium and hedges
  - the (random) P&L experienced on a delta-hedged derivative position

- 2nd portion of Theta is interesting info

- Useful Θ is obtained by adjusting \( \frac{dP}{dt}\delta t \) for financing of premium, deltas:

\[
Θ = \left( \frac{dP}{dt} - rP + (r - q)S \frac{dP}{dS} \right)
\]

\[
= \frac{1}{\delta t} \left( e^{-r\delta t}P(t + \delta t, Se^{(r-q)\delta t}) - P(t, S) \right)
\]
Theta – 2

- With Θ thus defined:

\[ P \& L = \Theta \delta t + \frac{1}{2} \frac{d^2 P}{dS^2} \delta S^2 \]

\[ = \frac{1}{2} \frac{S^2 d^2 P}{dS^2} \left( \frac{\delta S^2}{S^2} - \hat{\sigma}^2 \delta t \right) \] in Black-Scholes

- We’re not done yet
  - Adjust Θ for coupons paid/received during \([t, t + \delta t]\)

- More basic issue: numerical accuracy of Θ estimate
  - Daily Θ is typically of the order of a few bps
  - Monte Carlo schemes started at \(t, t + \delta t\) run on different schedules: \(P(t)\) poor control variate of \(P(t + \delta t)\)

\[ \frac{dP}{dt} = \frac{P(t + \delta t, S) - P(t, S)}{\delta t} = \text{white noise. With adjustments: whiter noise} \]

- Can we compute Θ differently?

- Actually – which Theta should we calculate? Come back to this later
Standard Theta – 1

- Std Theta: price at $t + \delta t$ keeping $\hat{\sigma}_{KT}$ unchanged
  - Rationale: so that Theta of vanilla = BS Theta

- Intuition: $\Theta \equiv$ effect of vol/vol-of-vol on $[t, t + \delta t]$
- To get $\Theta$, turn vol off on $[t, t + \delta t]$ for 2nd pricing
- OK In BS case – does not work in $\sigma(t)$ BS or LV model
- Still basic idea of 2 prices at $t$ is OK

- Can $e^{-r\delta t} P(t + \delta t, S e^{(r-q)\delta t}, \hat{\sigma}_{KT})$ be computed as a time-$t$ price?
  - Find vols $\hat{\sigma}_{KT}^*$ such that:
    \[
    P(t, S, \hat{\sigma}_{KT}^*) = e^{-r\delta t} P(t + \delta t, S e^{(r-q)\delta t}, \hat{\sigma}_{KT})
    \]
    - $\hat{\sigma}_{KT}^*$ given by:
      \[
      (\hat{\sigma}_{KT_i}^*)^2 T = \hat{\sigma}_{KT_i}^2 (T_i - \delta t) \quad \Rightarrow \quad \hat{\sigma}_{KT_i}^* = \hat{\sigma}_{KT_i} \sqrt{\frac{T_i - \delta t}{T_i}}
      \]
      - $\Theta$ given by:
        \[
        \Theta = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^*) - P(t, S, \hat{\sigma}_{KT}) \right]
        \]
Standard Theta – 2

- Effective vols $\hat{\sigma}_{KT_i}^*$ generate adequate scenario $\hat{\sigma}_{KT_i}$: $\hat{\sigma}_{KT_i}(t + \delta t) = \hat{\sigma}_{KT_i}(t)$

- What about implied vols for intermediate maturities?
  - Assume affine interpolation of total variance for given moneyness
  - For $T \in [T_1, T_2]$, $\hat{\sigma}_T$ is given by:
    \[
    T\hat{\sigma}_T^2 = T_1\hat{\sigma}_1^2 + \frac{\hat{\sigma}_2^2 T_2 - \hat{\sigma}_1^2 T_1}{T_2 - T_1} (T - T_1)
    \]
  - In standard technique, reprice at $t + \delta t$ with same $\hat{\sigma}_{KT_i}$ $\Rightarrow$ $\hat{\sigma}_{KT}$ is changed
  - Same interpolation scheme applied to $\hat{\sigma}_{KT_i}^*$ volatilities produces same time-$t + \delta t$ value for $\hat{\sigma}_{KT}$
  - OK

- What about maturities $T < T_1$?
  - Scenario generated by $\hat{\sigma}_{KT_i}^*$ $\neq$ scenario of standard technique
  - For $T = \delta t$, $\hat{\sigma}_{K\delta t}^*$ should be zero and it’s not

$\Rightarrow$ Insert artificial maturity $T = \delta t$ in market smile
$\Rightarrow$ Takes care of all issues
$\Rightarrow$ Theta of vanillas of maturities $\geq \delta t \equiv$ std Theta
Standard Theta - 3

- Use $\hat{\sigma}_{KT}^2$ rather than $\hat{\sigma}_{KT}$

$$\Theta = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^2) - P(t, S, \hat{\sigma}_{KT_i}^2) \right]$$

$$= \frac{1}{\delta t} \left[ P(t, S, \left(1 - \frac{\delta t}{T_i}\right)\hat{\sigma}_{KT_i}^2) - P(t, S, \hat{\sigma}_{KT_i}^2) \right]$$

- We don’t need to set $\hat{\sigma}_{KT} = \delta t$ to zero:

$$\Theta = \frac{1}{\epsilon \delta t} \left[ P\left(t, S, \left(1 - \frac{\epsilon \delta t}{T_i}\right)\hat{\sigma}_{KT_i}^2\right) - P\left(t, S, \hat{\sigma}_{KT_i}^2\right) \right]$$

- Theta can also be computed as a centered difference:

$$\Theta = \frac{1}{2\delta t} \left[ P\left(t, S, \left(1 - \frac{\delta t}{T_i}\right)\hat{\sigma}_{KT_i}^2\right) - P\left(t, S, \left(1 + \frac{\delta t}{T_i}\right)\hat{\sigma}_{KT_i}^2\right) \right]$$

- If $\delta t$ is small (i.e. shorter than derivative’s 1st spot observation) then above expression does not depend on choice of $\delta t$:

$$\Theta = - \sum \frac{\hat{\sigma}_{KT_i}^2}{T_i} \frac{dP}{d(\hat{\sigma}_{KT_i}^2)}$$
Standard Theta – recipe

Recipe:

- Insert artificial maturity $\delta t$ in smile
- Insert it also in original smile, for numerical stability
- Compute Theta as:

$$\Theta = \frac{1}{\epsilon \delta t} \left[ P\left(t, S, \left(1 - \frac{\epsilon \delta t}{T_i}\right) \tilde{\sigma}_{KT_i}^2\right) - P\left(t, S, \tilde{\sigma}_{KT_i}^2\right) \right]$$

- Typically take $\delta t = 5$ days, $\epsilon = 0.5$
- To get exact same numerical value as usual theta set $\delta t = 1$ day, $\epsilon = 1$

Do we really need to insert the artificial $\delta t$ maturity in the market smile?

- If no spot observation before $T_1$, no need
- In tests below no artificial maturity
Standard Theta – multi-asset derivatives

- Works just the same: calculate 2nd time-$t$ price with $\hat{\sigma}_{KT_i}^*$ volatilities

- Need to insert artificial maturity $\delta t$
  - Even when no spot observation prior to $T_1$
  - Marginal distributions for $T \geq T_1$ do not depend on smiles for maturities $< T_1$, but joint densities do

- What if correlations are re-calibrated when repricing at $t + \delta t$?
  - Do same: recalibrate correlations
  - Compute 2nd time-$t$ price using recalibrated correlations

- What if instead of applying the $\hat{\sigma}_{KT} \rightarrow \hat{\sigma}_{KT_i}^*$ shift for all assets at once, we compute each asset’s individual contribution?
  - Get Theta as a sum of individual contributions: $\Theta = \sum_i \Theta_i$
  - In BS model, for European payoff, $\Theta_i$ is related to Vega$_i$:

$$\Theta_i = -\frac{1}{2} \sum_j \Gamma_{ij} S_i S_j \rho_{ij} \sigma_i \sigma_j = -\frac{1}{2T} \sigma_i \frac{dP}{d\sigma_i}$$
Standard Theta – Local-stochastic volatility models

- Part of time value is generated by vol of stoch vol
- 1st time-\( t \) price computed normally: LV component calibrated on market smile
- For 2nd time-\( t \) price:
  - Use \( \hat{\sigma}_{KT} \) vol surface
  - Calibrate LV component, setting vol-of-vol to zero on time interval \([t, t + \delta t]\)
  - Price at \( t \) in LSV model, using zero vol-of-vol on \([t, t + \delta t]\)
Let’s test this on Vanillas

- Use LV model
- Use smooth smile generated by P1 model with SX5E-like parameters – no repo, no divs.
  - So that standard method least at disadvantage

- Pdate = 20 Jan 17. First mat = 17 March 17. Last mat = 18 Dec 26

- Have used same (good) MC engine in both sets of tests
Standard technique – 16k MC paths – Pricing date = 20-Jan-2017
▶ Minus 1-day Theta, Spot = 100, strikes 50 to 200
New technique – 16k MC paths – Pricing date = 20-Jan-2017

- Minus 1-day Theta, Spot = 100, strikes 50 to 200. $\delta t = 5$ days, $\epsilon = 0.5$.
Conclusion

- Works better than std method – especially for longer maturities
  - New technique more immune to choice of time discretization

- What if there are cash dividends?
  - Can’t use simple formula for $\hat{\sigma}_{KT}$ anymore
  - (a) Price undiscounted vanillas with pricing date $= \delta t$/spot $= \text{Fwd}(S, \text{mat} = \delta t)$
  - (b) Back out vol with pricing date $= 0$/spot $= S$

- What if implied vol scenario is different (FX, commodities): $\hat{\sigma}_{KT}$ not fixed?
  - Do same thing: price vanillas at $\delta t$ with chosen scenario for $\hat{\sigma}_{KT}$, then back out vols $\hat{\sigma}_{KT}^*$ at $t = 0$

- Test performed on smooth payoffs. Does it work with barrier options as well – or autocalls? See 2nd talk

- Basic question not addressed yet
  - Theta scenario: keep implied vols for fixed strikes/maturities unchanged
  - Generates a real number which we call Theta
  - Is this number useful? How to calculate a (more) useful Theta?
Two ideal types of derivative traders

- Vanilla trader
  - risk-manages vanillas
  - hedge instruments: underlyings/futures/forwards
  - Theta/Gamma P&L

- Exotics trader
  - risk-manages exotics
  - hedge instruments: underlyings/futures/forwards + vanillas/delta-hedged vanillas
  - Theta/Gamma-Vanna-Volga P&L
Useful Theta – Black-Scholes

- Useful Theta = breakeven of gammas = impact of vol during $\delta t$
  - Reprice at $t$ but turn vols off during $[t, t + \delta t]$.

- In mono/multi BS model with fixed vol this indeed yields $\Theta$

- $\Theta \neq 0$ is a sign that hedge is not static and needs to be rebalanced ($\Gamma \neq 0$)

- $\Theta$ of static linear combination of hedge instruments (spot, forward) = 0

- Move on to exotics
  - Hedge instruments: spot + vanilla options
  - Model takes as input market prices of hedge instruments & generates hedge ratios
  - Natural criterion: $\Theta$ of static linear combination of hedge instruments = zero
  - $\Theta \neq 0$: deltas not stable $\Rightarrow$ Gamma/Volga/Vanna P&L
Useful Theta – exotics

- Models for exotics take as inputs market prices of hedge instruments
  - $S +$ vanilla prices (or implied vols)
    \[ P(t, S, O_{KT}) \text{ or } P(t, S, \sigma_{KT}) \]
  - Vanillas traditionally called ”vega hedges” – but they’re just deltas

- LV is such a model: generates deltas on $S$ and $O_{KT}$
- Assume $r = q = 0$. Calculate $\Theta_{\text{exotic}}$ just as we calculate standard $\Theta$ – keeping prices of hedge instruments fixed:
  \[
  \Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ P(t + \delta t, S, O_{KT}) - P(t, S, O_{KT}) \right]
  \]

- $\Theta_{\text{exotic}}$ of static basket of hedge instruments (spot & vanilla options) $\equiv$ zero

- Fixed option prices $\Rightarrow$ scenarios for $\sigma_{KT}$?
- What does $\Theta_{\text{exotic}}$ measure?
Useful Theta – exotics – 2

- \[ \delta O_{KT} = 0 \Rightarrow \delta \hat{\sigma}_{KT} \text{ such that:} \]
  \[
  (\hat{\sigma}_{KT} + \delta \hat{\sigma}_{KT})^2(T - \delta t) = \hat{\sigma}_{KT}^2 T
  \]

- Here \( r = q = 0 \) but formula for \( \delta \hat{\sigma}_{KT} \) holds for \( r \neq 0, q \neq 0 \). \( \Theta_{\text{exotic}} \) given by:
  \[
  \Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ P(t + \delta t, S, \hat{\sigma}_{KT} + \delta \hat{\sigma}_{KT}) - P(t, S, \hat{\sigma}_{KT}) \right]
  \]
  \[
  \delta \hat{\sigma}_{KT} = \frac{\delta t}{2T} \hat{\sigma}_{KT}
  \]

- In LV model, P&L of delta/vega-hegded position – including recalibration of LV function:
  \[
  \left[ P(t + \delta t, S + \delta S, O_{KT} + \delta O_{KT}) - P(t, S, O_{KT}) \right] - \frac{dP}{dS} \delta S - \sum_{KT} \frac{dP}{dO_{KT}} \delta O_{KT}
  \]
  \[
  = \frac{1}{2} \frac{d^2P}{dS^2} \left( \delta S^2 - \sigma^2(t, S) S^2 \delta t \right)
  \]
  \[
  + \sum_{KT} \frac{d^2P}{dSdO_{KT}} \left( \delta S \delta O_{KT} - \sigma^2(t, S) S^2 \frac{dO_{KT}}{dS} \delta t \right)
  \]
  \[
  + \frac{1}{2} \sum_{KT, K'T'} \frac{d^2P}{dO_{KT}dO_{K'T'}} \left( \delta O_{KT} \delta O_{K'T'} - \sigma^2(t, S) S^2 \frac{dO_{KT}}{dS} \frac{dO_{K'T'}}{dS} \delta t \right)
  \]
Useful Theta – exotics – 3

- $O_{KT}(t, S)$: price of $K$, $T$ vanilla in the LV model – evaluated with the local volatility function $\sigma(t, S)$ calibrated to $K$, $T$ smile

- Set $\delta S = \delta O_{KT} = 0$ in expression of P&L to get $\Theta_{\text{exotic}}$

  $\Rightarrow \Theta_{\text{exotic}}$ indeed measures cost of dynamic rehedging: Gammas, Vannas, Volgas

  $\Rightarrow \Theta_{\text{exotic}} \neq 0$: signal that hedge ratios are not static, i.e. derivative is exotic:

$$
\frac{d}{dS} \left( \frac{d\mathcal{P}}{dS} \right) \neq 0, \quad \frac{d}{dO_{KT}} \left( \frac{d\mathcal{P}}{dO_{KT}} \right) \neq 0, \quad \frac{d}{dS} \left( \frac{d\mathcal{P}}{dO_{KT}} \right) \neq 0
$$

- With rate/repo:

$$
\Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ e^{-r\delta t} \mathcal{P}(t + \delta t, Se^{(r-q)\delta t}, O_{KT}e^{r\delta t}) - \mathcal{P}(t, S, O_{KT}) \right]
$$

- Comment: in most LSV models, P&L of delta & vega hedged position does not break down into matching Gamma/Theta contributions, unlike in LV model\(^1\)

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\(^1\)P&L leakage unless model obeys simple condition – see *Local stochastic volatility: models and non-models*: http://www.lorenzobergomi.com
Why not use $\hat{\sigma}_{KT}$ rather than $O_{KT}$ in P&L expansion?

- European payoffs have non-zero Vanna/Volga
  - No indicators of convexity, since can be statically hedged with vanillas

- Assume hedge instrument whose price is $S$ (think vanilla option)
  - We represent its value with variable $z$ (think implied vol): $S(z, t)$.
  - Consider (exotic) derivative. Price $P(z, t)$
  - Parametrization-independent measure of convexity? $\frac{d^2P}{dz^2}$ not adequate, as $\frac{d^2S}{dz^2} \neq 0$.
  - Carry P&L incurred only if hedge ratio varies with $S$

- Parametrization-independent measure of convexity: variation of hedge ratios wrt prices of hedge instruments
  - Hedge ratio $\Delta = \frac{dP}{dz} \left(\frac{dS}{dz}\right)^{-1}$
    $$\frac{d\Delta}{dS} = \left(\frac{dS}{dz}\right)^{-1} \frac{d}{dz} \left[ \frac{dP}{dz} \left(\frac{dS}{dz}\right)^{-1} \right] = \left(\frac{dS}{dz}\right)^{-2} \left[ \frac{d^2P}{dz^2} - \Delta \frac{d^2S}{dz^2} \right]$$

  $$P&L = \Delta \delta S + \frac{1}{2} \frac{d\Delta}{dS} \delta S^2$$

- Conclusion: $\frac{d^2P}{dSd\hat{\sigma}_{KT}}$, $\frac{d^2P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{K'T'}}$ no indicators of actual risk/convexity

- $\frac{d^2P}{dSdO_{KT}}$, $\frac{d^2P}{dO_{KT}dO_{K'T'}}$ are. In sequel called ”Vanna”, ”Volga”
How to compute $\Theta_{\text{exotic}}$ using time-$t$ prices only?

In LV model
- Insert artificial maturity $\delta t$ in market smile and set $\tilde{\sigma}_{K\delta t} = 0$
- Turning spot vol off on $[0, \delta t]$ also turns vols of vols off

In LSV models
- Insert additional short maturity $\delta t$ in market smile and set $\tilde{\sigma}_{K\delta t} = 0$
- Turn vols of vols off on $[0, \delta t]$

Compute $\Theta_{\text{exotic}}$ as:

$$\Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ P(t, S, \tilde{\sigma}^\dagger_{KT}) - P(t, S, \tilde{\sigma}_{KT}) \right]$$

with

$$\tilde{\sigma}^\dagger_{K,\delta t} = 0$$

$$\tilde{\sigma}^\dagger_{KT_i} = \tilde{\sigma}_{KT_i} \text{ for } T_i > \delta t$$

$\Theta_{\text{exotic}}$ quantifies exoticty of derivative
- As priced by model
- With flat input smile, in LV model, vols of vols $\equiv 0 \Rightarrow \Theta_{\text{exotic}} = 0$
Splitting Theta

- Calculate $\Theta$ and $\Theta_{\text{exotic}}$ using only time-$t$ prices

- Standard Theta $\Theta$: Vega risk analysis (RA) with $\hat{\sigma}_{KT}^*$ vols

$$\Theta = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^*) - P(t, S, \hat{\sigma}_{KT}) \right]$$

- Exotic Theta $\Theta_{\text{exotic}}$: Vega RA with $\hat{\sigma}_{KT}^{\dagger}$ vols

$$\Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^{\dagger}) - P(t, S, \hat{\sigma}_{KT}) \right]$$

- If book $\equiv$ portfolio of vanilla options/risks, $\Theta_{\text{exotic}} = 0$

- Imagine book has large (standard) theta
  - Maybe it’s just a large static portfolio of vanillas $\Rightarrow$ zero convexity: can be statically hedged, nothing to worry about
  - Or maybe this Theta is generated by sizeable Gammas/Vannas/Volgas

- How to tell?
Splitting Theta – 2

► Answer: split $\Theta$ into two pieces: $\hat{\sigma}_{KT} \rightarrow \hat{\sigma}_{KT}^\dagger \rightarrow \hat{\sigma}_{KT}^*$

$$\Theta = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^*) - P(t, S, \hat{\sigma}_{KT}) \right] = \Theta_{\text{vanilla}} + \Theta_{\text{exotic}}$$

$$\Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^\dagger) - P(t, S, \hat{\sigma}_{KT}) \right]$$

$$\Theta_{\text{vanilla}} = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}_{KT}^*) - P(t, S, \hat{\sigma}_{KT}^\dagger) \right]$$

$\Rightarrow \Theta_{\text{vanilla}} = \text{sum of BS Thetas of the perfect vanilla hedge portfolio:}$

$$\Theta_{\text{vanilla}} = \frac{1}{\delta t} \sum_{K,T,\delta t} \frac{dP}{d\hat{\sigma}_{KT}} (\hat{\sigma}_{K,T}^* - \hat{\sigma}_{K,T}^\dagger) = \frac{1}{\delta t} \sum_{K,T} \frac{dP}{d\hat{\sigma}_{KT}} (\hat{\sigma}_{K,T}^* - \hat{\sigma}_{K,T})$$

$$= - \sum_{K,T} \frac{dP}{d\hat{\sigma}_{KT}} \frac{\hat{\sigma}_{K,T}}{2T} = \sum_{K,T} \frac{dP}{d\hat{\sigma}_{KT}} \frac{\Theta_{K,T}}{\text{Vega}_{KT}^{BS}}$$

$$= \sum_{K,T} n_{KT} \Theta_{K,T}^{BS}$$

$n_{KT} = \frac{dP}{d\hat{\sigma}_{KT}} \frac{1}{\text{Vega}_{KT}^{BS}}$: number of $K, T$ vanilla options in vega hedge

► $\Theta_{\text{vanilla}} \neq 0 \Rightarrow \text{book is not properly Vega-hedged}$
Recap

- New technique for computing standard Theta: one additional price with slightly different implied vols.

- Standard Theta can be broken down into $\Theta_{\text{vanilla}}$ and $\Theta_{\text{exotic}}$ by calculating one additional price:
  - $\Theta_{\text{vanilla}} = \text{sum of standard thetas of full Vega hedge portfolio}$
  - $\Theta_{\text{exotic}} = \text{measure of convexity of derivative}$
  - $\Theta_{\text{exotic}} = \text{sum of real Gamma, Vanna & Vammas times model-generated break-even levels}$

- Example of breakdown of Theta of autocall in 2nd talk.

- We’re now able to compute $\Theta_{\text{exotic}}$. What should we be doing next?
Computing realized Gamma, Vanna & Volga P&L

- Consider overnight moves of spot, implied vols $\delta S$, $\delta \hat{\sigma}_{KT}$

- P&L of exotic derivative using implied vols – zero repo/interest rate:

$$P&L = \left( \frac{dP}{dS} \delta S + \sum \frac{dP}{d\hat{\sigma}} \delta \hat{\sigma} \right) + (\Theta_{\text{Vanilla}} + \Theta_{\text{Exotic}}) \delta t$$

$$+ \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 + \sum \frac{d^2P}{dSd\hat{\sigma}} \delta S \delta \hat{\sigma} + \frac{1}{2} \sum \frac{d^2P}{d\hat{\sigma}d\hat{\sigma}'} \delta \hat{\sigma} \delta \hat{\sigma}'$$

where $\hat{\sigma}$ stands for $\hat{\sigma}_{KT}$ and $\hat{\sigma}'$ for $\hat{\sigma}_{K'T'}$

- Valid accounting expression – except last 2 contributions cannot be interpreted as real Gamma/Vanna/Volga P&Ls: $\neq 0$ even for a vanilla option

- Use prices rather than implied vols – zero repo/interest rate:

$$P&L = \left( \frac{dP}{dS} \delta S + \sum \frac{dP}{dO} \delta O \right) + \Theta_{\text{Exotic}} \delta t$$

$$+ \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 + \sum \frac{d^2P}{dSdO} \delta S \delta O + \frac{1}{2} \sum \frac{d^2P}{dOdO'} \delta O \delta O'$$

- Last 3 pieces real Gamma/Vanna/Volga P&Ls – how to compute them practically??
Computing realized Gamma, Vanna & Volga P&L – 2

- Want to compute realized Gamma/Vanna/Volga P&L:

\[ P&L_{GVV} = \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 + \sum_O \frac{d^2P}{dSdO} \delta S \delta O + \frac{1}{2} \sum_{O,O'} \frac{d^2P}{dOdO'} \delta O \delta O' \]

- If \( \delta S, \delta O \) small, given by:

\[ P&L_{GVV} = \frac{1}{2} \left[ P(S + \delta S, O + \delta O) + P(S - \delta S, O - \delta O) \right] - P(S, O) \]

- Pricing libs use implied vols, not prices. Which implied vol scenarios to use?

- Overnight realized moves of spot, implied vols: \( \delta S, \hat{\sigma}_{KT} \)

- For each \((K, T)\) vanilla, overnight variation of option price is \( \delta O(\delta S, \hat{\sigma}) \)

- Find \( \hat{\sigma}^*_{KT} \) such that \((-\delta S, -\hat{\sigma}^*)\) generates opposite price variation:

\[ \delta O(-\delta S, -\hat{\sigma}^*) = -\delta O(\delta S, \hat{\sigma}) \]

\[ \text{i.e.} \]

\[ \left( O(S - \delta S, \hat{\sigma} - \hat{\sigma}^*) - O(S, \hat{\sigma}) \right) = -\left( O(S + \delta S, \hat{\sigma} + \hat{\sigma}^*) - O(S, \hat{\sigma}) \right) \]
Computing realized Vanna & Volga P&L – 3

Total Gamma/Vanna/Volga P&L is given by:

\[
P&L_{GVV} = \frac{1}{2} \left[ P(S + \delta S, \hat{\sigma} + \delta \hat{\sigma}) + P(S - \delta S, \hat{\sigma} - \delta \hat{\sigma}^*) \right] - P(S, \hat{\sigma}) \]

\[P&L_{GVV} = 0\] if derivative is statically hedgeable with underlying & vanillas

How to compute separate Volga/Vanna P&L contributions?

More basic question – before even worrying about 2nd order P&Ls:

Total realized P&L of book is \( P(S + \delta S, \hat{\sigma} + \delta \hat{\sigma}) - P(S, \hat{\sigma}) \)

How much of this is Delta/Vega directional P&L??

Easy - no need for the full Vega \((K, T)\) map: use same \(\delta \hat{\sigma}^*\) and calculate:

\[
P&L_{\Delta} = \frac{1}{2} \left[ P(S + \delta S, \hat{\sigma} + \delta \hat{\sigma}) - P(S - \delta S, \hat{\sigma} - \delta \hat{\sigma}^*) \right] \]

\[P&L_{\Delta} \neq 0 \implies \text{large directional contribution to realized P&L}\]

Book is not properly \((K, T)\) Vega-hedged
Computing $P&L_{\text{Volga}}$

\[ P&L_{\text{Volga}} = \frac{1}{2} \left[ \mathcal{P}(S, O + \delta O) + \mathcal{P}(S, O - \delta O) \right] - \mathcal{P}(S, O) \]

- Scenario of implied vols is $\delta \hat{\sigma}_{KT}$
- For each $(K, T)$ vanilla, define vol shift $\delta \hat{\sigma}_{KT}$ such that $\delta O(-\delta \hat{\sigma}) = -\delta O(\delta \hat{\sigma})$

\[ O(S, \hat{\sigma} - \delta \hat{\sigma}) - O(S, \hat{\sigma}) = -\left( O(S, \hat{\sigma} + \delta \hat{\sigma}) - O(S, \hat{\sigma}) \right) \]

- Volga P&L given by:

\[ P&L_{\text{Volga}} = \frac{1}{2} \left[ P(S, \hat{\sigma} + \delta \hat{\sigma}) + P(S, \hat{\sigma} - \delta \hat{\sigma}) \right] - P(S, \hat{\sigma}) \]

- $P&L_{\text{Volga}} = 0$ for vanillas

Can be used to calculate a Volga-fee adjustment for a derivatives book

- Select scenario $\delta \hat{\sigma}_{KT}$
- Adjustment is given by $P&L_{\text{Volga}}$
- Anything vanilla-like (payoff or risk) in the book is unaffected
Existence of $\delta^*$ and of $\delta^-$?

- $\delta^*/\delta^-$ not guaranteed to exist. Ex for $\delta^-$: $\delta O(-\delta^-) = -\delta O(\delta^*)$
- What if $\delta O > O$? Then $O - \delta O$ is negative??

Imagine $S$ moves from $S = 2$ to $S = 5$. How to calculate realized Gamma P&L?

- Realized Gamma P&L is: $\frac{1}{2} \frac{d^2 f}{dS^2} \delta S^2$
- If $\delta S$ small, then Gamma P&L given by: $\frac{1}{2} (f(S + \delta S) + f(S - \delta S) - 2f(S))$

What if $\delta S$ is large?

- Calculate 2nd order derivative using smaller variation $\epsilon \delta S$:
  
  $$\frac{d^2 f}{dS^2} = \frac{f(S + \epsilon \delta S) + f(S - \epsilon \delta S) - 2f(S)}{2(\epsilon \delta S)^2}$$

- Gamma P&L is given by:
  
  $$\frac{1}{2} \frac{d^2 f}{dS^2} \delta S^2 = \frac{f(S + \epsilon \delta S) + f(S - \epsilon \delta S) - 2f(S)}{2(\epsilon \delta S)^2} \delta S^2$$
  
  $$= \frac{1}{\epsilon^2} \cdot \frac{1}{2} (f(S + \epsilon \delta S) + f(S - \epsilon \delta S) - 2f(S))$$

Recipe: (a) rescale spot/vol variation by $\epsilon$, (b) compute P&L with rescaled variations, (c) expand result by $\frac{1}{\epsilon^2}$

In practice choose largest value of $\epsilon$ that works in most situations
Computing $P\&L_{Vanna}$

$$P\&L_{Vanna} = \frac{1}{2} \left[ P(S + \delta S, O + \delta O) + P(S - \delta S, O - \delta O) \right] - \frac{1}{2} \left[ P(S + \delta S, O - \delta O) + P(S - \delta S, O + \delta O) \right]$$

- Joint spot/vol scenario is: $\delta S, \delta \hat{\sigma}_{KT}$

- For each $(K, T)$ vanilla, define 3 implied volatility variations: $\delta \hat{\sigma}_{KT}^{--}, \delta \hat{\sigma}_{KT}^{+-}, \delta \hat{\sigma}_{KT}^{-+}$ such that:
  \begin{align*}
  \delta O(-\delta S, -\delta \hat{\sigma}^{--}) &= -\delta O(\delta S, \delta \hat{\sigma}) \\
  \delta O(\delta S, -\delta \hat{\sigma}^{+-}) &= -\delta O(\delta S, \delta \hat{\sigma}) \\
  \delta O(-\delta S, \delta \hat{\sigma}^{-+}) &= \delta O(\delta S, \delta \hat{\sigma})
  \end{align*}

- Vanna P&L/adjustment obtained by calculating 4 prices:

$$P\&L_{Vanna} = \frac{1}{2} \left[ P(S + \delta S, \hat{\sigma} + \delta \hat{\sigma}) + P(S - \delta S, \hat{\sigma} - \delta \hat{\sigma}^{--}) \right] - \frac{1}{2} \left[ P(S + \delta S, \hat{\sigma} - \delta \hat{\sigma}^{+-}) + P(S - \delta S, \hat{\sigma} + \delta \hat{\sigma}^{-+}) \right]$$

- $P\&L_{Vanna} = 0$ for vanillas
Conclusion

- $\Theta$ efficiently calculated using 2 time-$t$ prices with $\hat{\sigma}_{KT}$ and $\hat{\sigma}^*_KT$

- $\Theta$ splits into $\Theta_{\text{vanilla}} + \Theta_{\text{exotic}}$
  - Cost: compute one extra time-$t$ price with $\hat{\sigma}^\dagger_{KT}$
  - $\Theta_{\text{exotic}} \neq 0 \implies$ book carries exotics Vanna/Volga risks
  - $\Theta_{\text{vanilla}} \neq 0 \implies$ book is not properly Vega hedged

- Simple method for calculating realized Gamma/Vanna/Volga P&Ls
  ... and for calculating realized Delta/Vega P&L

- Can be used to calculate Volga/Vanna fee adjustment – that leaves vanilla risks untouched

- Can do quite a lot with well-chosen global Vega risk analyses