The local volatility model

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Global Derivatives – Amsterdam, May 2015
Intro – things heard on the street

- LV model used inconsistently: local vol surface is calibrated today; only to be recalibrated tomorrow. 
  ⇒ violates model’s assumption of fixed LV surface.

- Trading practice: don’t use LV delta – instead compute ”sticky-strike” delta: move \( S \), keep implied vols unchanged, recalibrate local vol surface.
  - Rationale: so that vanilla options have BS delta.

- On a scale from dirty to downright ugly, where do we stand?
  - What is the carry P&L of an option position?
  - By the way, what’s the delta of a vanilla option?
Outline

- Usable models?
- The carry P&L of the LV model
- The delta – the delta of a vanilla option
- Break-even levels for vols of implied vols / covariance of spot and implied volatilities
- Conclusion

⇒ Material of this presentation part of forthcoming book
Practically usable market models

▷ Define assets $A_i$ to be used as hedging instruments – pricing function $P(t, A_1, \ldots, A_n)$

▷ First order contribution to P&L cancelled by delta hedging:

$$P\&L = -[P(t + \delta t, A + \delta A) - P(t, A)] + \sum \frac{dP}{dA_i} (\delta A_i - \mu_i A_i \delta t)$$

▷ First non-trivial contribution to P&L – order 1 in $\delta t$, 2 in $\delta A$:

$$P\&L = -\frac{dP}{dt} \delta t - \frac{1}{2} \sum \frac{d^2P}{dA_i dA_j} \delta A_i \delta A_j$$

▷ Condition for non-nonsensical P&L: there exists a positive breakeven covariance matrix $C_{ij}(t, A)$ such that:

$$\frac{dP}{dt} = \frac{1}{2} \sum \frac{d^2P}{dA_i dA_j} A_i A_j C_{ij}$$

$$P\&L = -\frac{1}{2} \sum A_i A_j \frac{d^2P}{dA_i dA_j} \left( \frac{\delta A_i}{A_i} \frac{\delta A_j}{A_j} - C_{ij} \delta t \right)$$

▷ Trademark of a market model
▷ Sign of P&L depends on mismatch between model & realized covariances

▷ Ideally we would like to set the $C_{ij}$ – otherwise may need to use implied levels.
The local volatility model

- In LV model $P^{LV}(t, S, \sigma)$. $\sigma(t, S)$ local volatility function.

- Use "local volatility delta" $\Delta^{LV} = \left. \frac{dP^{LV}}{dS} \right|_{\sigma}$. From pricing equation, P&L during $\delta t$:

$$P^{LV} = -\frac{1}{2} S^2 \frac{d^2P^{LV}}{dS^2} \left( \left( \frac{\delta S}{S} \right)^2 - \sigma^2(t, S) \delta t \right)$$

- In LV model – with fixed LV function – implied vols functions of $t, S$:

$$\hat{\sigma}_{KT}(t, S) \equiv \Sigma^{LV}_{KT}(t, S, \sigma)$$

- $P^{LV}$ actual P&L only if market implied vols move as prescribed by $\Sigma^{LV}_{KT}(t, S, \sigma)$.

  $\Rightarrow \Delta^{LV}$ useless

- Pricing function $P(t, S, \hat{\sigma}_{KT})$. LV function $\sigma$ calibrated on vanilla smile:

$$P(t, S, \hat{\sigma}_{KT}) \equiv P^{LV}(t, S, \sigma [t, S, \hat{\sigma}_{KT}])$$

$$P^{LV}(t, S, \sigma) = P(t, \Sigma^{LV}_{KT}(t, S, \sigma))$$

- Let’s compute the carry P&L of an option position.
Carry P&L with the LV model – (black-box) pricing function $P(t, S, \hat{\sigma}_{KT})$

- Start with P&L of *naked* option position during $\delta t$:

$$P&L = - [P(t + \delta t, S + \delta S, \hat{\sigma}_{KT} + \delta \hat{\sigma}_{KT}) - (1 + r\delta t)P(t, S, \hat{\sigma}_{KT})]$$

- Expanding at order one in $\delta t$ and two in $\delta S$ and $\delta \hat{\sigma}_{KT}$:

$$P&L = rP\delta t$$

$$- \frac{dP}{dt} \delta t - \frac{dP}{dS} \delta S - \frac{dP}{d\hat{\sigma}_{KT}} \bullet \delta \hat{\sigma}_{KT}$$

$$- \left( \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 + \frac{d^2P}{dS d\hat{\sigma}_{KT}} \bullet \delta \hat{\sigma}_{KT} \delta S + \frac{1}{2} \frac{d^2P}{d\hat{\sigma}_{KT} d\hat{\sigma}'_{KT'}} \bullet \delta \hat{\sigma}_{KT} \delta \hat{\sigma}'_{KT'} \right)$$

- Notation $\bullet$ stands for:

$$\frac{df}{d\hat{\sigma}_{KT}} \bullet \delta \hat{\sigma}_{KT} \equiv \int \int dKdT \frac{\delta f}{\delta \hat{\sigma}_{KT}} \delta \hat{\sigma}_{KT} \equiv \Sigma_{ij} \frac{df}{d\hat{\sigma}_{K_i T_j}} \delta \hat{\sigma}_{K_i T_j}$$

- $\frac{dP}{dS}$, $\frac{dP}{dt}$ are computed keeping the $\hat{\sigma}_{KT}$ fixed -- the LV function is *not* fixed.

- Define sticky-strike delta $\Delta^{SS}$:

$$\Delta^{SS} = \left. \frac{dP}{dS} \right|_{\hat{\sigma}_{KT}}$$
Carry P&L with the LV model – 2

- Utilize that $P$ is given by LV pricing equation:

$$P^{LV}(t, S, \sigma) = P\left(t, S, \hat{\sigma}_{KT} = \Sigma^{LV}_{KT}(t, S, \sigma)\right)$$

- Express derivatives of $P^{LV}$ in terms of derivatives of $P$:

$$\frac{dP^{LV}}{dt} = \frac{dP}{dt} + \frac{dP}{d\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dt}$$

$$\frac{dP^{LV}}{dS} = \frac{dP}{dS} + \frac{dP}{d\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS}$$

$$\frac{d^2P^{LV}}{dS^2} = \left(\frac{d^2P}{dS^2} + 2 \frac{d^2P}{dSd\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} + \frac{d^2P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{KT'}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} \cdot \frac{d\Sigma^{LV}_{KT'}}{dS}\right) + \frac{dP}{d\hat{\sigma}_{KT}} \cdot \frac{d^2\Sigma^{LV}_{KT}}{dS^2}$$

- Now insert these expressions in LV pricing equation:

$$\frac{dP^{LV}}{dt} + (r - q)S\frac{dP^{LV}}{dS} + \frac{1}{2}\sigma^2(t, S)S^2\frac{d^2P^{LV}}{dS^2} = rP^{LV}$$

- ... to generate relationship involving derivatives of $P$. 


Yields:

\[
\frac{dP}{dt} = rP - (r - q)S \frac{dP}{dS} - \frac{dP}{d\hat{\sigma}_{KT}} \cdot \mu_{KT} \\
- \frac{1}{2} \sigma^2(t, S) S^2 \left( \frac{d^2P}{dS^2} + 2 \frac{d^2P}{dSd\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} + \frac{d^2P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{KT'}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} \frac{d\Sigma^{LV}_{KT'}}{dS} \right)
\]

with \( \mu_{KT} \) given by:

\[
\mu_{KT} = \frac{d\Sigma^{LV}_{KT}}{dt} + \frac{1}{2} \sigma^2(t, S) S^2 \frac{d^2\Sigma^{LV}_{KT}}{dS^2} + (r - q)S \frac{d\Sigma^{LV}_{KT}}{dS}
\]

Now rewrite P&L of \textit{naked} option position in slide 6:

\[
P\&L = - \frac{dP}{dS} (\delta S - (r - q)S \delta t) - \frac{dP}{d\hat{\sigma}_{KT}} \cdot (\delta \hat{\sigma}_{KT} - \mu_{KT} \delta t) \\
+ \frac{1}{2} \sigma^2(t, S) S^2 \left( \frac{d^2P}{dS^2} + 2 \frac{d^2P}{dSd\hat{\sigma}_{KT}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} + \frac{d^2P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{KT'}} \cdot \frac{d\Sigma^{LV}_{KT}}{dS} \frac{d\Sigma^{LV}_{KT'}}{dS} \right) \delta t \\
- \left( \frac{1}{2} \frac{d^2P}{dS^2} \delta S^2 + \frac{d^2P}{dSd\hat{\sigma}_{KT}} \cdot \delta \hat{\sigma}_{KT} \delta S + \frac{1}{2} \frac{d^2P}{d\hat{\sigma}_{KT}d\hat{\sigma}_{KT'}} \cdot \delta \hat{\sigma}_{KT} \delta \hat{\sigma}_{KT'} \right)
\]
Carry P&L with the LV model – 4

- Introduce implied (log-normal) vol of vol of \( \hat{\sigma}_{KT} \):

\[
\nu_{KT} = \frac{1}{\Sigma_{KT}^{LV}} \frac{d\Sigma_{KT}^{LV}}{dS} S \sigma(t, S)
\]

- Rewrite P&L as:

\[
P&L = - \frac{dP}{dS} (\delta S - (r - q)S \delta t) - \frac{dP}{d\hat{\sigma}_{KT}} \bullet (\delta \hat{\sigma}_{KT} - \mu_{KT} \delta t)
\]

\[
- \frac{1}{2} S^2 \frac{d^2 P}{dS^2} \left[ \frac{\delta S^2}{S^2} - \sigma^2(t, S) \delta t \right]
\]

\[
- \frac{d^2 P}{dS \, d\hat{\sigma}_{KT}} \bullet S \hat{\sigma}_{KT} \left[ \frac{\delta S}{S} \frac{\delta \hat{\sigma}_{KT}}{\hat{\sigma}_{KT}} - \sigma(t, S) \nu_{KT} \delta t \right]
\]

\[
- \frac{1}{2} \frac{d^2 P}{d\hat{\sigma}_{KT} \, d\hat{\sigma}_{K'T'}} \bullet \hat{\sigma}_{KT} \hat{\sigma}_{K'T'} \left[ \frac{\delta \hat{\sigma}_{KT}}{\hat{\sigma}_{KT}} \frac{\delta \hat{\sigma}_{K'T'}}{\hat{\sigma}_{K'T'}} - \nu_{KT} \nu_{K'T'} \delta t \right]
\]

- Only uses market observables: \( P(t, S, \hat{\sigma}_{KT}) \) – no LV function involved.

- P&L expression is that of market model.
  - Variance/covariance breakeven levels are well-defined + make up a positive covariance matrix.
  - Delta is sticky-strike delta \( \frac{dP}{dS} \), vegas simple vegas.
Carry P&L with the LV model – 5

- $\hat{\sigma}_{KT} \equiv$ implied vol plays no special role. Use instead price $O_{KT}$: $P(t, S, O_{KT}).$

\[ P(t, S, \hat{\sigma}_{KT}) = P(t, S, O_{KT} = P_{KT}^{BS}(t, S, \hat{\sigma}_{KT})) \]

\[ P^{LV}(t, S, \sigma) = P(t, S, \Omega^{LV}_{KT}(t, S, \sigma)) \]

$\Omega^{LV}_{KT}(t, S, \sigma)$ price in LV model with LV function $\sigma$.

- Drift $\mu_{KT}$ simplifies:

\[ \mu_{KT} = \frac{d\Omega^{LV}_{KT}}{dt} + \frac{1}{2} \sigma^2(t, S) S^2 \frac{d^2\Omega^{LV}_{KT}}{dS^2} + (r - q)S \frac{d\Omega^{LV}_{KT}}{dS} = r\Omega^{LV}_{KT} = rO_{KT} \quad \text{OK} \]

- P&L of naked option position – using now asset prices – cf slide 4:

\[ P\&L = - \frac{dP}{dS} (\delta S - (r - q)S \delta t) - \frac{dP}{dO_{KT}} \cdot (\delta O_{KT} - rO_{KT} \delta t) \]

\[ - \frac{1}{2} \frac{d^2P}{dS^2} \left[ \delta S^2 - \sigma^2(t, S) S^2 \delta t \right] \]

\[ - \frac{d^2P}{dSdO_{KT}} \cdot \left[ \delta S \delta O_{KT} - \sigma^2(t, S) S^2 \frac{d\Omega^{LV}_{KT}}{dS} \delta t \right] \]

\[ - \frac{1}{2} \frac{d^2P}{dO_{KT}dO_{KT}'} \cdot \left[ \delta O_{KT} \delta O_{KT}' - \sigma^2(t, S) S^2 \frac{d\Omega^{LV}_{KT}}{dS} \frac{d\Omega^{LV}_{KT}'}{dS} \delta t \right] \]
Carry P&L with the LV model – conclusion

- P&L expression has typical form of market models – OK
- Hedging instruments all treated on equal footing.
- Implied break-even levels of cross-gammas are payoff-independent – are determined by market smile prevailing at time $t$.
  - spot/vol correl = $-100\%$
  - vol/vol correl = $100\%$
  - vol of $\tilde{\sigma}_{KT}$ is $\nu_{KT} = \frac{1}{\Sigma_{KT}} \frac{d\Sigma_{KT}}{dS} S\sigma(t, S)$
- Hedge ratios are simply $\left. \frac{dP}{dS} \right|_{O_{KT}}$ and $\left. \frac{dP}{dO_{KT}} \right|_{S}$
- Delta of the local volatility model is $\Delta^{MM} = \left. \frac{dP}{dS} \right|_{O_{KT}}$.
- Delta of vanilla option is an irrelevant notion.
  - akin to asking model to generate a hedge ratio of one hedging instrument on another hedging instrument.
- Result seems $\approx$ natural, but see 2nd talk this afternoon for strange pathologies in local/stoch vol models.
So, what is the LV model?

- The LV model is a market model for the underlying and vanilla options ... that happens to have a 1-d Markov representation in terms of $(t, S)$.

- This is a mathematical technicality – of which the LV function is a by-product – that facilitates pricing. Nothing fundamental.

- Daily recalibration of LV function is exactly how it has to be used.

- Consequences of 1-d Markov representation:
  - The break-even covariance matrix is of rank 1 – correls $= 100\%$.
  - No control on break-even levels of volatilities of implied volatilities. They are set by the configuration of $S$, $\hat{\sigma}_{KT}$ and will vary unpredictably.
    - Like them, use model; don’t like them, don’t use model.

- This is how much we can get in a model with a 1-d Markov representation.
Consistency of sticky-strike and market-model deltas

- Use $S, O_{KT} \Rightarrow P(t, S, O_{KT})$. Hedge ratios $\Delta^{MM} = \frac{dP}{dS} \bigg|_{O_{KT}}, \quad \frac{dP}{dO_{KT}} \bigg|_{S}$
- Use $S, \hat{\sigma}_{KT} \Rightarrow P(t, S, \hat{\sigma}_{KT})$. Hedge ratios $\Delta^{SS} = \frac{dP}{dS} \bigg|_{\hat{\sigma}_{KT}}, \quad \frac{dP}{d\hat{\sigma}_{KT}} \bigg|_{S}$
  - $\frac{dP}{d\hat{\sigma}_{KT}}$ offset by trading $BS$-delta-hedged vanilla options

- Hedge portfolio is:
  $$\Pi = \frac{dP}{dS} S + \frac{dP}{dO_{KT}} \bullet O_{KT}$$

- Rewrite in terms of delta-hedged vanillas:
  $$\Pi = \left[ \frac{dP}{dS} + \frac{dP}{dO_{KT}} \bullet \frac{dP^{BS}_{KT}}{dS} \right] S + \frac{dP}{dO_{KT}} \bullet \left[ O_{KT} - \frac{dP^{BS}_{KT}}{dS} S \right]$$

- Spot hedge ratio?
  - Move spot + move vanilla prices by their Black-Scholes deltas
    akin to: move vanilla prices keeping implied vols fixed $\Rightarrow$ sticky strike delta
  $$\Delta^{SS} = \frac{dP}{dS} + \frac{dP}{dO_{KT}} \bullet \frac{dP^{BS}_{KT}}{dS}$$

- Once hedge portfolio broken down into underlying + naked vanilla options, delta
  always equal to $\Delta^{MM} = \frac{dP}{dS} \bigg|_{O_{KT}}$.

- Nothing fundamental about $\Delta^{SS}$ – tied to a particular representation of vanilla option prices.
Dynamics in LV model

▶ What’s left before we can use LV model? Output the $\nu_{KT}$, see if we like them.
  ▶ More practical to look at implied vols for floating strike – fixed moneyness.

▶ Look at vols of vols and spot/vol covariances.

▶ For ATMF vol equivalently look at SSR $R_T$

$$R_T = \frac{1}{S_T} \frac{\langle d\hat{\sigma}_{FT} d\ln S \rangle}{\langle (d\ln S)^2 \rangle} = \frac{1}{S_T} \frac{d\hat{\sigma}_{FT}}{d\ln S}$$

$$S_T = \left. \frac{d\hat{\sigma}_{KT}}{d\ln K} \right|_{F_T}$$

$$\text{vol}(\hat{\sigma}_{FT}) = R_T \left( S_T \frac{\hat{\sigma}_{F0}}{\hat{\sigma}_{FT}} \right)$$

▶ Assume following expression for LV function:

$$\sigma(t, S) = \sigma(t) + \alpha(t)x + \frac{\beta(t)}{2}x^2, \quad x = \ln \left( \frac{S}{F_t} \right)$$

and calculate $S_T$, $R_T$ at order 1 in $\alpha(t), \beta(t)$. 
Dynamics in LV model – 2

- Use variance \( u = \sigma^2 \) and write \( u(t, S) = u_0(t) + \delta u(t, S) \). At order 1 in \( \delta u \):

\[
\hat{\sigma}^2_{KT} = \frac{1}{T} \int_0^T dt \int_{-\infty}^{+\infty} dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \ u(t, F_t e^{\frac{\omega_t}{\sqrt{T}} x_K + \frac{\sqrt{(\omega_T - \omega_t)\omega_t}}{\sqrt{T}} y})
\]

where \( x_K = \ln(\frac{K}{F_T}) \) and \( \omega_t = \int_0^t u_0(\tau) d\tau \).

- Expanding around a cst \( \sigma(t) = \sigma_0 \): \( u_0 = \sigma_0^2 \)

\[
\hat{\sigma}_{KT} = \frac{1}{T} \int_0^T dt \int_{-\infty}^{+\infty} dy \frac{e^{-y^2/2}}{\sqrt{2\pi}} \ \sigma(t, F_t e^{\frac{t}{\sqrt{T}} x_K + \sigma_0 \sqrt{(T-t) t} y})
\]

- From this get:

\[
S_T = \left. \frac{d\hat{\sigma}_{KT}}{d \ln K} \right|_{K=F_T} = \frac{1}{T} \int_0^T \frac{t}{T} \alpha(t) dt \quad \text{"skew averaging" – V. Piterbarg}
\]

\[
\left. \frac{d^2\hat{\sigma}_{KT}}{d \ln K^2} \right|_{K=F_T} = \frac{1}{T} \int_0^T \left( \frac{t}{T} \right)^2 \beta(t) dt
\]

\[
\left. \frac{d\hat{\sigma}_{KT}}{d \ln S} \right|_{K=F_T} = \frac{1}{T} \int_0^T \left( 1 - \frac{t}{T} \right) \alpha(t) dt
\]
Dynamics in LV model – 3

From 1st equation: \( \alpha(t) = \frac{d}{dt}(tS_t) + S_t \).

\[
\frac{d\hat{\sigma}_{FT(S)T}}{d\ln S} = \left( \frac{d\hat{\sigma}_{KT}}{d \ln K} \bigg|_{K=F_T} + \frac{d\hat{\sigma}_{KT}}{d \ln S} \bigg|_{K=F_T} \right) = \frac{1}{T} \int_0^T \alpha(t) dt = S_T + \frac{1}{T} \int_0^T S_t dt
\]

Get expression of SSR: \( R_T = \frac{1}{S_T} \frac{\langle d\hat{\sigma}_{FT} d\ln S \rangle}{\langle (d\ln S)^2 \rangle} = \frac{1}{S_T} \frac{d\hat{\sigma}_{FT(S)T}}{d\ln S} : \)

\[
R_T = 1 + \frac{1}{T} \int_0^T \frac{S_t}{S_T} dt
\]

Limiting behavior

short maturities:

\[
\lim_{T \to 0} R_T = 2 \quad \text{OK}
\]

long maturities – take \( S_T \propto \frac{1}{T^\gamma} \):

\[
\lim_{T \to \infty} R_T = \frac{2 - \gamma}{1 - \gamma}
\]

For typical value \( \gamma = \frac{1}{2} \), \( \lim_{T \to \infty} R_T = 3 \).
What about expanding around a deterministic volatility $\sigma_0(t)$ rather than a cst $\sigma_0$?

$$S_T = \left. \frac{d\hat{\sigma}_{KT}}{d\ln K} \right|_{K=F_T} = \frac{1}{T} \int_0^T \frac{\hat{\sigma}_t^2}{\hat{\sigma}_T^2} \frac{\sigma_0(t)}{\hat{\sigma}_T} \alpha(t) dt$$

$$\left. \frac{d\hat{\sigma}_{KT}}{d\ln S} \right|_{K=F_T} = \frac{1}{T} \int_0^T \left(1 - \frac{\hat{\sigma}_t^2}{\hat{\sigma}_T^2} \right) \frac{\sigma_0(t)}{\hat{\sigma}_T} \alpha(t) dt$$

$$\left. \frac{d\hat{\sigma}_{FTT}}{d\ln S} \right|_{K=F_T} = S_T + \frac{1}{T} \int_0^T \frac{\sigma_0^2(t)}{\hat{\sigma}_T \hat{\sigma}_T} S_t dt$$

where $\hat{\sigma}_t = \sqrt{\frac{1}{t} \int_0^t \sigma_0^2(u) du}$.

SSR given by:

$$R_T = 1 + \frac{1}{T} \int_0^T \left( \frac{\sigma_0^2(t)}{\hat{\sigma}_T \hat{\sigma}_T} \right) \frac{S_t}{S_T} dt$$

Better approx of $R_T$ for strongly sloping term structures of ATMF volatilities.
Dynamics in LV model – 5

- Check approx of SSR on 2 smiles of Eurostoxx50

![Graphs showing smiles and term structures](image)

**Figure:** Top: smiles of the Eurostoxx50 index for a maturity ≃ 1 year observed on October 4, 2010 (left) and May 16, 2013 (right). Bottom: term structures of ATMF skew and power-law fits with $\gamma = 0.37$ (left), $\gamma = 0.52$ (right), as a function of $T$ (years).
Dynamics in LV model – 6

Real versus approximate SSR

Figure: $R_T$ as a function of $T$ (years) computed: (a) in FD (actual), (b) using expression $R_T = 1 + \frac{1}{T} \int_0^T \frac{S_t}{S_T} dt$ (approx).

What about smile with $S_T \propto \frac{1}{T}$? Approx fomula gives $\lim_{T \to \infty} R_T = \infty$ (logarithmic divergence of $R_T$):

Approx slightly overestimates SSR.
Conclusion

▶ LV model is a genuine market model for underlying + vanilla options ... that happens to possess a 1-d Markov representation in terms of (t,S).

▶ Generates well-defined break-even levels for spot/vol and vol/vol covariances in the carry P&L.

▶ Daily recalibration of LV function – an ancillary object – is exactly how model should be used and deltas calculated.
  ▶ Spot/vol break-even correlations $= -100\%$, vol/vol break-even correlations $= 100\%$.
  ▶ Volatilities of implied volatilities given by: $\text{vol}(\hat{\sigma}_{KT}) = \frac{1}{\Sigma_{KT}^{LV}} \frac{d\Sigma_{KT}^{LV}}{dS} S\sigma(t, S)$.

▶ Delta is well-defined: $\Delta^{MM} = \frac{dP}{dS}\bigg|_{O_{KT}}$. Delta of vanilla option irrelevant notion.

▶ When vega-hedging with (BS) delta-hedged vanilla options, sticky-strike delta should be used.

▶ Good approximate formula for sizing up break-even vols of ATMF vols – or equivalently SSR:

$$R_T = 1 + \frac{1}{T} \int_0^T \frac{S_t}{S_T} dt$$

$$\text{vol}(\hat{\sigma}_{FT_T}) = R_T \left( S_T \frac{\hat{\sigma}_{F0}}{\sigma_{FT_T}} \right)$$