Correlations in Asynchronous Markets

Lorenzo Bergomi

Global Markets Quantitative Research
SOCIETE GENERALE
Corporate & Investment Banking
lorenzo.bergomi@sgcib.com

Global Derivatives, Paris, April 2011

Paper available on www.ssrn.com

Outline

- Motivation
- Estimating correlations and volatilities in asynchronous markets
- Historical correlations: Stoxx50 – S&P500 – Nikkei
- Comparison with other heuristic estimators – options and correlation swaps
- Correlations larger than 1
- Conclusion

Motivation

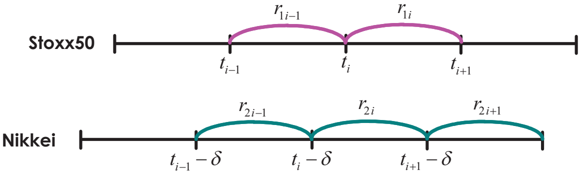
- Equity derivatives generally involve baskets of stocks/indices traded in different geographical areas
- Operating hours of: Asian and European exchanges, Asian and American exchanges, usually have no overlap

- Standard methodology on equity derivatives desks:
  - Use standard multi-asset model based on assumption of continuously traded securities
  - Compute/trade deltas at the close of each market, using *stale* values for securities not trading at that time
  - Likewise, valuation is done using *stale* values for securities whose markets are closed

▷ **How should we estimate volatility and correlation parameters ?**

Correlation estimation in asynchronous markets

- Consider following situation:



  - Valuation of the option is done at the close of the Stoxx50
  - Deltas are computed and traded on the market close of each security

- Daily P&L:

$$\begin{aligned} P\&L &= -\left[f\left(t_{i+1},\ S_{1,i+1},\ S_{2,i+1}\right) - f\left(t_i,\ S_{1,i},\ S_{2,i}\right)\right] \\ &\quad + \frac{df}{dS_1}\left(t_i,\ S_{1,i},\ S_{2,i}\right)\left(S_{1,i+1} - S_{1,i}\right) \\ &\quad + \frac{df}{dS_2}\left(t_i - \delta,\ S_{1,i-1},\ S_{2,i}\right)\left(S_{2,i+1} - S_{2,i}\right) \end{aligned}$$

- Note that arguments of $\frac{df}{dS_2}$ are different

- Rewrite delta on S_2 so that arguments are same as f and $\frac{df}{dS_1}$

$$\frac{df}{dS_2}\left(t_i-\delta,\ S_{1,i-1},\ S_{2,i}\right) = \frac{df}{dS_2}\left(t_i,\ S_{1,i},\ S_{2,i}\right) - \frac{d^2 f}{dS_1 dS_2}\left(t_i,\ S_{1,i},\ S_{2,i}\right)\left(S_{1,i}-S_{1,i-1}\right)$$

- Other correction terms contribute at higher order in Δ
- P&L now reads:

$$\begin{aligned} P\&L &= -\left[f\left(t_{i+1},\ S_{1,i+1},\ S_{2,i+1}\right) - f\left(t_i,\ S_{1,i},\ S_{2,i}\right)\right] \\ &\quad + \frac{df}{dS_1}\left(S_{1,i+1}-S_{1,i}\right) + \left[\frac{df}{dS_2} - \frac{d^2 f}{dS_1 dS_2}\left(S_{1,i}-S_{1,i-1}\right)\right]\left(S_{2,i+1}-S_{2,i}\right) \end{aligned}$$

- Expanding at 2nd order in δS_1, δS_2:

$$P\&L = -\frac{df}{dt}\Delta - \left[\frac{1}{2}\frac{d^2 f}{dS_1^2}\delta S_{1+}^2 + \frac{1}{2}\frac{d^2 f}{dS_1^2}\delta S_{2+}^2 + \frac{d^2 f}{dS_1 dS_2}\delta S_{1+}\delta S_{2+}\right] - \frac{d^2 f}{dS_1 dS_2}\delta S_{1-}\delta S_{2+}$$

- Assume f is given by a Black-Scholes equation:

$$\frac{df}{dt} + \frac{\sigma_1^2}{2} S_1^2 \frac{d^2 f}{dS_1^2} + \frac{\sigma_2^2}{2} S_2^2 \frac{d^2 f}{dS_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{d^2 f}{dS_1 dS_2} = 0$$

- P&L now reads:

$$\begin{aligned} P\&L &= -\frac{1}{2} S_1^2 \frac{d^2 f}{dS_1^2} \left[\left(\frac{\delta S_1^+}{S_1} \right)^2 - \sigma_1^2 \Delta \right] - \frac{1}{2} S_2^2 \frac{d^2 f}{dS_2^2} \left[\left(\frac{\delta S_2^+}{S_2} \right)^2 - \sigma_2^2 \Delta \right] \\ & \quad - S_1 S_2 \frac{d^2 f}{dS_1 dS_2} \left[\left(\frac{\delta S_1^-}{S_1} + \frac{\delta S_1^+}{S_1} \right) \frac{\delta S_2^+}{S_2} - \rho \sigma_1 \sigma_2 \Delta \right] \end{aligned}$$

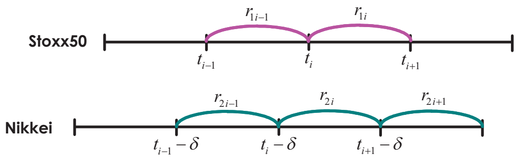
▷ Prescription for estimating volatilities & correlations so that P&L vanishes on average:

$$\sigma_1^{\star 2} = \frac{1}{\Delta} \left\langle \left(\frac{\delta S_1^+}{S_1} \right)^2 \right\rangle \quad \sigma_2^{\star 2} = \frac{1}{\Delta} \left\langle \left(\frac{\delta S_2^+}{S_2} \right)^2 \right\rangle \quad \rho^\star \sigma_1^\star \sigma_2^\star = \frac{1}{\Delta} \left\langle \left(\frac{\delta S_1^-}{S_1} + \frac{\delta S_1^+}{S_1} \right) \frac{\delta S_2^+}{S_2} \right\rangle$$

  ▷ **Volatility estimators are the usual ones, involving daily returns**
  ▷ **The correlation estimator involves daily returns as well**

- Define $r_i = \frac{\delta S_i^+}{S_i}$. At lowest order in Δ, $\frac{\delta S_i^-}{S_i} \simeq \frac{\delta S_i^-}{S_{i-1}}$

$$\sigma_1^{\star 2} = \frac{1}{\Delta}\left\langle r_{1i}^2 \right\rangle \qquad \sigma_2^{\star 2} = \frac{1}{\Delta}\left\langle r_{2i}^2 \right\rangle \qquad \rho^\star = \frac{\left\langle \left(r_{1i-1} + r_{1i}\right) r_{2i} \right\rangle}{\sqrt{\left\langle r_{1i}^2 \right\rangle \left\langle r_{2i}^2 \right\rangle}}$$



- Had we chosen the close of the Nikkei for valuing the option: symmetrical estimator:

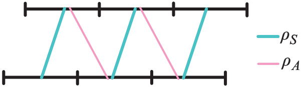
$$\sigma_1^{\star 2} = \frac{1}{\Delta}\left\langle r_{1i}^2 \right\rangle \qquad \sigma_2^{\star 2} = \frac{1}{\Delta}\left\langle r_{2i}^2 \right\rangle \qquad \rho^\star = \frac{\left\langle r_{1i} \left(r_{2i} + r_{2i+1}\right) \right\rangle}{\sqrt{\left\langle r_{1i}^2 \right\rangle \left\langle r_{2i}^2 \right\rangle}}$$

- If returns are time-homogeneous $\left\langle r_{1i-1} r_{2i} \right\rangle = \left\langle r_{1i} r_{2i+1} \right\rangle$
  In practice $\frac{1}{N}\sum_1^N \left(r_{1i-1} + r_{1i}\right) r_{2i} - r_{1i}\left(r_{2i} + r_{2i+1}\right) = \frac{1}{N}\left(r_{10} r_{21} - r_{1N} r_{2N+1}\right)$

▷ Difference between two estimators of $\rho^\star$: finite size effect of order $\frac{1}{N}$

- In conclusion, in asynchronous markets: 2 correlations ρ_S, ρ_A:

$$\rho_S = \frac{\langle r_{1i}\ r_{2i}\rangle}{\sqrt{\langle r_{1i}^2\rangle\langle r_{2i}^2\rangle}} \quad \rho_A = \frac{\langle r_{1i}\ r_{2i+1}\rangle}{\sqrt{\langle r_{1i}^2\rangle\langle r_{2i}^2\rangle}}$$



and derivatives should be priced with $\rho^\star$:

$$\rho^\star = \rho_S + \rho_A$$

- ▷ Does $\rho^\star$ depend on the particular delta strategy used in derivation ?
- ▷ Is $\rho^\star$ in $[-1, 1]$?
- ▷ How does $\rho^\star$ compare to standard correlations estimators evaluated with 3-day, 5-day, n-day returns ?

- What if had computed deltas differently – for example "predicting" the value of the stock not trading at the time of computation ?
  - ▷ Option delta-hedged one way minus option delta-hedged the other way. Final P&L is:

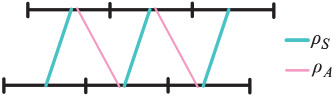
$$\sum(\Delta_t^a - \Delta_t^b)(S_{t+\Delta} - S_t)$$

  - ▷ Price of pure delta strategy is zero: **correlation estimator is independent on delta strategy used in derivation**

- Imagine processes are continuous yet observations are asynchronous: assume that $\rho\sigma_1\sigma_2$, σ_1^2, σ_2^2 are periodic functions with period $\Delta = 1$ day:

$$\rho_S = \frac{\frac{1}{\Delta}\int_t^{t+\Delta-\delta}\rho\sigma_1\sigma_2\ ds}{\sqrt{\frac{1}{\Delta}\int_t^{t+\Delta}\sigma_1^2 ds}\sqrt{\frac{1}{\Delta}\int_{t-\delta}^{t+\Delta-\delta}\sigma_2^2 ds}} \qquad \rho_A = \frac{\frac{1}{\Delta}\int_{t+\Delta-\delta}^{t+\Delta}\rho\sigma_1\sigma_2\ ds}{\sqrt{\frac{1}{\Delta}\int_t^{t+\Delta}\sigma_1^2 ds}\sqrt{\frac{1}{\Delta}\int_{t-\delta}^{t+\Delta-\delta}\sigma_2^2 ds}}$$

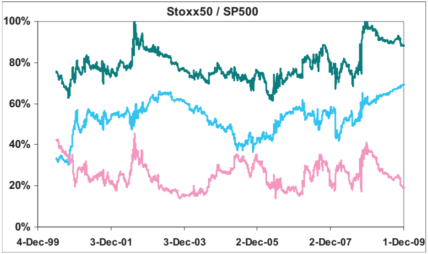
$$\rho^\star = \rho_S + \rho_A$$

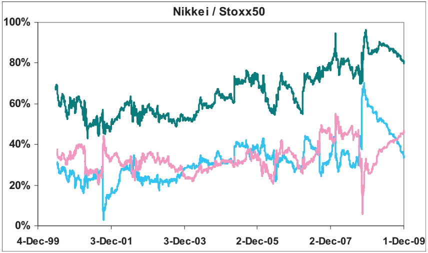


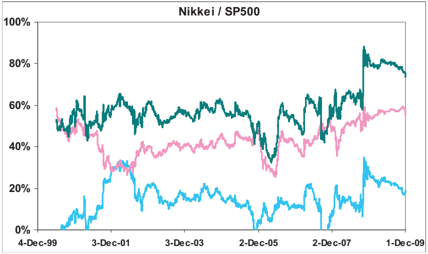
▷ Recovers value of "synchronous correlation": **no bias**

Historical correlations

- ρ_S (blue), ρ_A (pink), $\rho^\star = \rho_S + \rho_A$ (green) – 6-month EWMA







- ρ_S, ρ_A seem to move antithetically
  - Imagine $\sigma_1(s) = \sigma_1\lambda(s)$, $\sigma_2(s) = \sigma_2\lambda(s)$, ρ constant, with $\lambda(s)$ such that $\frac{1}{\Delta}\int_0^\Delta \lambda^2(s)ds = 1$. Then:

$$\rho_S = \rho\,\frac{1}{\Delta}\int_0^{\Delta-\delta}\lambda^2(s)\,ds$$
$$\rho_A = \rho\,\frac{1}{\Delta}\int_{\Delta-\delta}^{\Delta}\lambda^2(s)\,ds$$

  and $\rho^\star$ is given by:

$$\rho^\star = \rho_S + \rho_A = \rho$$

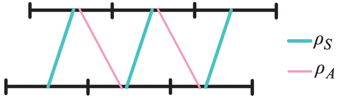
  - By changing $\lambda(s)$ we can change ρ_S, ρ_A, while $\rho^\star$ stays fixed.

▷ The relative sizes of ρ_S, ρ_A are given by the intra-day distribution of the realized covariance.

Comparison with heuristic estimators

- Trading desks have long ago realized that merely using ρ_S is inadequate
  - Standard fix: compute standard correlation using 3-day, 5-day, you-name-it, rather than daily returns
  - How do these estimators differ from $\rho^\star$?

- Connected issue: how do we price an n-day correlation swap ?



▷ An $n-$day correlation swap should be priced with ρ_n given by:

$$\rho_n = \rho_S + \frac{n-1}{n}\rho_A$$

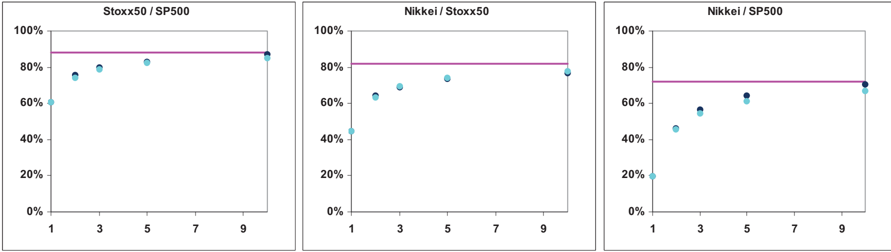
- For $n = 3$, $\rho_3 \;=\; \rho_S + \frac{2}{3}\rho_A$

- If no serial correlation in historical sample, standard correlation estimator applied to n-day returns yields ρ_n

Historical n-day correlations

- n-day correlations evaluated on 2004-2009 with:
  - n-day returns (dark blue)
  - using $\rho_S + \frac{n-1}{n}\rho_A$ (light blue)

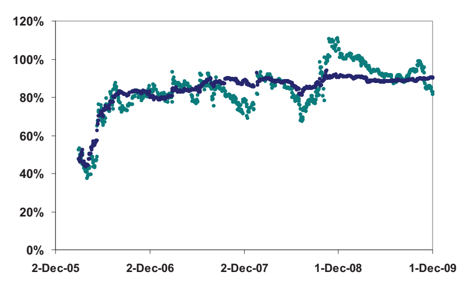
  compared to $\rho^\star$ (purple line)



- Common estimators ρ_3, ρ_5 underestimate $\rho^\star$

The S&P500 and Stoxx50 as synchronous securities

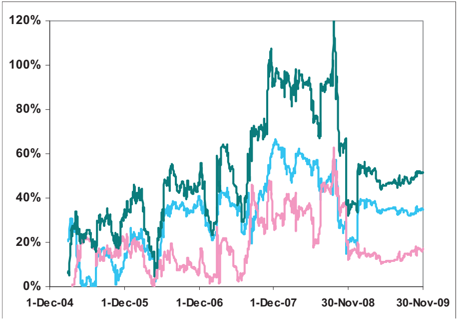
- European and American exchanges have some overlap. We can either:
  - delta-hedge asynchronously the S&P500 at 4pm New York time and the Stoxx50 at 5:30pm Paris time
  - delta-hedge simultaneously both futures at – say – 4pm Paris time
- 1st case: use $\rho^\star$, 2nd case: use standard correlation for synchronous securities – are they different ?
- $\rho^\star$ (light blue), standard sync. correlation (dark blue) – 3-month EWMA



- Matches well, but not identical: difference stems from residual *realized* serial correlations.

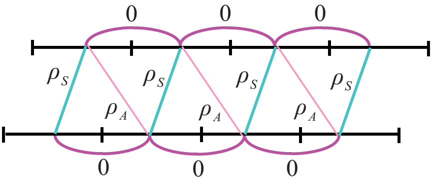
Correlations larger than 1

- Example of RBS/Citigroup correlations: ρ_S (blue), ρ_A (pink), $\rho^\star$ (green) – 3-month EWMA



- Are instances when $\rho^\star > 1$ an artifact ? Do they have financial significance ?

- Consider a situation when no serial correlation is present. The global correlation matrix is positive, by construction. How large can $\rho_S + \rho_A$ be ?



- Compute eigenvalues of full correlation matrix:
  - assume both ladder uprights consist of N segments, with periodic boundary conditions
  - assume eigenvalues have components $e^{ik\theta}$ on higher upright, $\alpha e^{ik\theta}$ on lower upright
  - express that λ is an eigenvalue:

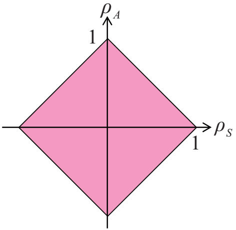
$$\begin{aligned} \alpha\rho_S + 1 + \alpha e^{i\theta} &= \lambda \\ \rho_S + \alpha + e^{-i\theta}\rho_A &= \lambda\alpha \end{aligned}$$

yields:

$$\lambda = 1 \pm \sqrt{(\rho_S + \rho_A \cos\theta)^2 + \rho_A^2 \sin^2\theta}$$

- Periodic boundary conditions impose $\theta = \frac{2n\pi}{N}$, where $n = 0 \ldots N-1$
- $\lambda(\theta)$ extremal for $\theta = 0, \pi$. For these values $\lambda = 1 \pm |\rho_S \pm \rho_A|$
- $\lambda > 0$ implies:

$$
\begin{aligned}
-1 &\leq \rho_S + \rho_A \leq 1 \\
-1 &\leq \rho_S - \rho_A \leq 1
\end{aligned}
$$



▷ **If no serial correlations** $\rho^\star \in [-1, 1]$

▷ Instances when $\rho^\star > 1$: evidence of serial correlations

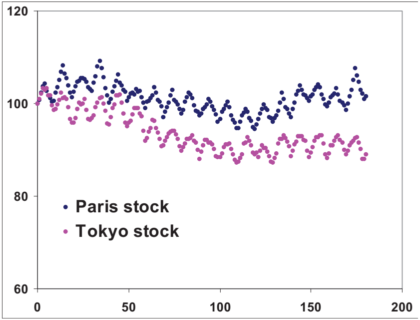
▷ **Impact of $\rho^\star > 1$ on trading desk: price with the right realized volatilities, 100% correlation → lose money !!**

Example with basket option

- Sell 6-month basket option on basket of Japanese stock & French stock.

  Payoff is $\left(\sqrt{\frac{S_1^T S_2^T}{S_1^0 S_2^0}} - 1\right)^+$

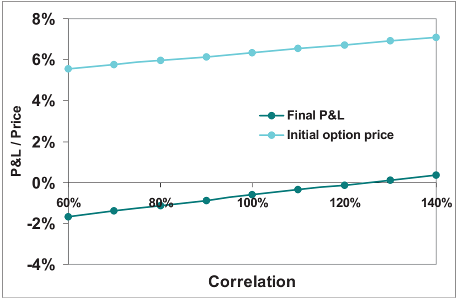
- Basket is lognormal with volatility given by $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2}$
- Use following "historical" data:



- Realized vols are 21.8% for S_1, 23.6% for S_2. Realized correlations are $\rho_S = 63.3\%$, $\rho_A = 57.6\%$: $\underline{\rho^\star = 121\%}$.

- Backtest delta-hedging of option with:
  - implied vols = realized vols
  - different implied correlations

- Initial option price and final P&L:



- Final P&L vanishes when one prices and risk-manages option with an implied correlation $\rho \approx 125\%$.

Conclusion

- It is possible to price and risk-manage options on asynchronous securities using the standard synchronous framework, provided special correlation estimator is used.
- Correlation estimator quantifies correlation that is materialized as cross-Gamma P&L.
- Same technique applies to asynchronous FX / Swap rates / ...
- Correlation swaps and options have to be priced with different correlations.
- Serial correlations may push realized value of $\rho^\star$ above 1: a short correlation position will lose money, even though one uses the right vols and 100% correlation.