Correlations in Asynchronous Markets

Lorenzo Bergomi

Global Markets Quantitative Research

SOCIETE GENERALE
Corporate & Investment Banking
lorenzo.bergomi@sgcib.com

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Outline

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- Estimating correlations and volatilities in asynchronous markets
- Historical correlations: Stoxx50 – S&P500 – Nikkei
- Comparison with other heuristic estimators – options and correlation swaps
- Correlations larger than 1
- Conclusion
Equity derivatives generally involve baskets of stocks/indices traded in different geographical areas.

Operating hours of: Asian and European exchanges, Asian and American exchanges, usually have no overlap.

Standard methodology on equity derivatives desks:
- Use standard multi-asset model based on assumption of continuously traded securities.
- Compute/trade deltas at the close of each market, using *stale* values for securities not trading at that time.
- Likewise, valuation is done using *stale* values for securities whose markets are closed.

How should we estimate volatility and correlation parameters?
Consider following situation:

Valuation of the option is done at the close of the Stoxx50
Deltas are computed and traded on the market close of each security

Daily P&L:

\[
P&L = - [f(t_{i+1}, S_{1,i+1}, S_{2,i+1}) - f(t_i, S_{1,i}, S_{2,i})] \\
+ \frac{df}{dS_1}(t_i, S_{1,i}, S_{2,i})(S_{1,i+1} - S_{1,i}) \\
+ \frac{df}{dS_2}(t_i - \delta, S_{1,i-1}, S_{2,i})(S_{2,i+1} - S_{2,i})
\]

Note that arguments of \(\frac{df}{dS_2}\) are different
Rewrite delta on $S_2$ so that arguments are same as $f$ and $\frac{df}{dS_1}$

$$
\frac{df}{dS_2} (t_i - \delta, S_{1,i-1}, S_{2,i}) = \frac{df}{dS_2} (t_i, S_{1,i}, S_{2,i}) - \frac{d^2f}{dS_1 dS_2} (t_i, S_{1,i}, S_{2,i}) (S_{1,i} - S_{1,i-1})
$$

Other correction terms contribute at higher order in $\Delta$

P&L now reads:

$$
P&L = - [f (t_{i+1}, S_{1,i+1}, S_{2,i+1}) - f (t_i, S_{1,i}, S_{2,i})] + \frac{df}{dS_1} (S_{1,i+1} - S_{1,i}) + \left[ \frac{df}{dS_2} - \frac{d^2f}{dS_1 dS_2} (S_{1,i} - S_{1,i-1}) \right] (S_{2,i+1} - S_{2,i})
$$

Expanding at 2nd order in $\delta S_1, \delta S_2$:

$$
P&L = - \frac{df}{dt} \Delta - \left[ \frac{1}{2} \frac{d^2f}{dS_1^2} \delta S_{1+}^2 + \frac{1}{2} \frac{d^2f}{dS_2^2} \delta S_{2+}^2 + \frac{d^2f}{dS_1 dS_2} \delta S_{1+} \delta S_{2+} \right] - \frac{d^2f}{dS_1 dS_2} \delta S_{1-} \delta S_{2+}
$$
Assume $f$ is given by a Black-Scholes equation:

$$\frac{df}{dt} + \frac{\sigma_1^2}{2} S_1^2 \frac{d^2f}{dS_1^2} + \frac{\sigma_2^2}{2} S_2^2 \frac{d^2f}{dS_2^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{d^2f}{dS_1 dS_2} = 0$$

P&L now reads:

$$P&L = -\frac{1}{2} S_1^2 \frac{d^2f}{dS_1^2} \left[ \left( \frac{\delta S_1^+}{S_1} \right)^2 - \sigma_1^2 \Delta \right] - \frac{1}{2} S_2^2 \frac{d^2f}{dS_2^2} \left[ \left( \frac{\delta S_2^+}{S_2} \right)^2 - \sigma_2^2 \Delta \right]$$

$$- S_1 S_2 \left( \frac{d^2f}{dS_1 dS_2} \left[ \left( \frac{\delta S_1^-}{S_1} + \delta S_1^+ \right) \frac{\delta S_2^+}{S_2} - \rho \sigma_1 \sigma_2 \Delta \right] \right)$$

Prescription for estimating volatilities & correlations so that P&L vanishes on average:

$$\sigma_1^{*2} = \frac{1}{\Delta} \left\langle \left( \frac{\delta S_1^+}{S_1} \right)^2 \right\rangle \quad \sigma_2^{*2} = \frac{1}{\Delta} \left\langle \left( \frac{\delta S_2^+}{S_2} \right)^2 \right\rangle \quad \rho^{*} \sigma_1^{*} \sigma_2^{*} = \frac{1}{\Delta} \left\langle \left( \frac{\delta S_1^-}{S_1} + \delta S_1^+ S_1 \right) \frac{\delta S_2^+}{S_2} \right\rangle$$

Volatility estimators are the usual ones, involving daily returns

The correlation estimator involves daily returns as well
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Historical correlations
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- Define \( r_i = \frac{\delta S^+_i}{S_i} \). At lowest order in \( \Delta \), \( \frac{\delta S^-_i}{S_i} \approx \frac{\delta S^-_{i-1}}{S_{i-1}} \)

\[
\sigma^*_1 = \frac{1}{\Delta} \left\langle r_{1i}^2 \right\rangle \quad \sigma^*_2 = \frac{1}{\Delta} \left\langle r_{2i}^2 \right\rangle \quad \rho^* = \frac{\left\langle (r_{1i-1} + r_{1i}) r_{2i} \right\rangle}{\sqrt{\left\langle r_{1i}^2 \right\rangle \left\langle r_{2i}^2 \right\rangle}}
\]

- Had we chosen the close of the Nikkei for valuing the option: symmetrical estimator:

\[
\sigma^*_1 = \frac{1}{\Delta} \left\langle r_{1i}^2 \right\rangle \quad \sigma^*_2 = \frac{1}{\Delta} \left\langle r_{2i}^2 \right\rangle \quad \rho^* = \frac{\left\langle r_{1i} (r_{2i} + r_{2i+1}) \right\rangle}{\sqrt{\left\langle r_{1i}^2 \right\rangle \left\langle r_{2i}^2 \right\rangle}}
\]

- If returns are time-homogeneous \( \left\langle r_{1i-1} r_{2i} \right\rangle = \left\langle r_{1i} r_{2i+1} \right\rangle \)

In practice \( \frac{1}{N} \sum_{i=1}^{N} (r_{1i-1} + r_{1i}) r_{2i} - r_{1i} (r_{2i} + r_{2i+1}) = \frac{1}{N} (r_{10} r_{21} - r_{1N} r_{2N+1}) \)

\(\triangleright\) Difference between two estimators of \( \rho^* \): finite size effect of order \( \frac{1}{N} \)
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In conclusion, in asynchronous markets: 2 correlations $\rho_S$, $\rho_A$:

$$
\rho_S = \frac{\langle r_{1i} r_{2i} \rangle}{\sqrt{\langle r_{1i}^2 \rangle \langle r_{2i}^2 \rangle}} \quad \rho_A = \frac{\langle r_{1i} r_{2i+1} \rangle}{\sqrt{\langle r_{1i}^2 \rangle \langle r_{2i}^2 \rangle}}
$$

and derivatives should be priced with $\rho^*$:

$$
\rho^* = \rho_S + \rho_A
$$

- Does $\rho^*$ depend on the particular delta strategy used in derivation?
- Is $\rho^*$ in $[-1, 1]$?
- How does $\rho^*$ compare to standard correlations estimators evaluated with 3-day, 5-day, $n$-day returns?
What if had computed deltas differently – for example "predicting" the value of the stock not trading at the time of computation?

- Option delta-hedged one way minus option delta-hedged the other way. Final P&L is:
  \[
  \sum (\Delta^a_t - \Delta^b_t)(S_{t+\Delta} - S_t)
  \]

- Price of pure delta strategy is zero: correlation estimator is independent on delta strategy used in derivation

Imagine processes are continuous yet observations are asynchronous: assume that \(\rho \sigma_1 \sigma_2\), \(\sigma_1^2\), \(\sigma_2^2\) are periodic functions with period \(\Delta = 1\) day:

\[
\rho_S = \frac{\frac{1}{\Delta} \int_t^{t+\Delta-\delta} \rho \sigma_1 \sigma_2 \, ds}{\sqrt{\frac{1}{\Delta} \int_t^{t+\Delta} \sigma_1^2 \, ds} \sqrt{\frac{1}{\Delta} \int_t^{t+\Delta-\delta} \sigma_2^2 \, ds}}
\]

\[
\rho_A = \frac{\frac{1}{\Delta} \int_t^{t+\Delta} \rho \sigma_1 \sigma_2 \, ds}{\sqrt{\frac{1}{\Delta} \int_t^{t+\Delta} \sigma_1^2 \, ds} \sqrt{\frac{1}{\Delta} \int_t^{t+\Delta-\delta} \sigma_2^2 \, ds}}
\]

\[
\rho^* = \rho_S + \rho_A
\]

Recovers value of "synchronous correlation": no bias
Historical correlations

- $\rho_S$ (blue), $\rho_A$ (pink), $\rho^* = \rho_S + \rho_A$ (green) – 6-month EWMA
• $\rho_S, \rho_A$ seem to move antithetically
  
  Imagine $\sigma_1(s) = \sigma_1 \lambda(s)$, $\sigma_2(s) = \sigma_2 \lambda(s)$, $\rho$ constant, with $\lambda(s)$ such that 
  \[
  \frac{1}{\Delta} \int_0^{\Delta} \lambda^2(s) ds = 1.
  \]
  Then:
  
  \[
  \rho_S = \rho \frac{1}{\Delta} \int_0^{\Delta-\delta} \lambda^2(s) ds
  \]
  \[
  \rho_A = \rho \frac{1}{\Delta} \int_{\Delta-\delta}^{\Delta} \lambda^2(s) ds
  \]
  and $\rho^*$ is given by:
  
  \[
  \rho^* = \rho_S + \rho_A = \rho
  \]

• By changing $\lambda(s)$ we can change $\rho_S, \rho_A$, while $\rho^*$ stays fixed.

▷ The relative sizes of $\rho_S, \rho_A$ are given by the intra-day distribution of the realized covariance.
Comparison with heuristic estimators

- Trading desks have long ago realized that merely using $\rho_S$ is inadequate
  - Standard fix: compute standard correlation using 3-day, 5-day, you-name-it, rather than daily returns
  - How do these estimators differ from $\rho^*$?

- Connected issue: how do we price an $n$-day correlation swap?

An $n$–day correlation swap should be priced with $\rho_n$ given by:

$$\rho_n = \rho_S + \frac{n - 1}{n} \rho_A$$

- For $n = 3$, $\rho_3 = \rho_S + \frac{2}{3} \rho_A$

- If no serial correlation in historical sample, standard correlation estimator applied to $n$-day returns yields $\rho_n$
Historical \( n \)-day correlations

- \( n \)-day correlations evaluated on 2004-2009 with:
  - \( n \)-day returns (dark blue)
  - using \( \rho_S + \frac{n-1}{n} \rho_A \) (light blue)

  compared to \( \rho^* \) (purple line)

- Common estimators \( \rho_3, \rho_5 \) underestimate \( \rho^* \)
The S&P500 and Stoxx50 as synchronous securities

- European and American exchanges have some overlap. We can either:
  - delta-hedge asynchronously the S&P500 at 4pm New York time and the Stoxx50 at 5:30pm Paris time
  - delta-hedge simultaneously both futures at – say – 4pm Paris time

- 1st case: use $\rho^*$, 2nd case: use standard correlation for synchronous securities – are they different?

- $\rho^*$ (light blue), standard sync. correlation (dark blue) – 3-month EWMA

Matches well, but not identical: difference stems from residual *realized* serial correlations.
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Example of RBS/Citigroup correlations: $\rho_S$ (blue), $\rho_A$ (pink), $\rho^*$ (green) – 3-month EWMA

Are instances when $\rho^* > 1$ an artifact? Do they have financial significance?
Consider a situation when no serial correlation is present. The global correlation matrix is positive, by construction. How large can $\rho_S + \rho_A$ be?

Compute eigenvalues of full correlation matrix:

- assume both ladder uprights consist of $N$ segments, with periodic boundary conditions
- assume eigenvalues have components $e^{ik\theta}$ on higher upright, $\alpha e^{ik\theta}$ on lower upright
- express that $\lambda$ is an eigenvalue:

$$\alpha \rho_S + 1 + \alpha e^{i\theta} = \lambda$$
$$\rho_S + \alpha + e^{-i\theta} \rho_A = \lambda \alpha$$

yields:

$$\lambda = 1 \pm \sqrt{(\rho_S + \rho_A \cos \theta)^2 + \rho_A^2 \sin^2 \theta}$$
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- Periodic boundary conditions impose $\theta = \frac{2n\pi}{N}$, where $n = 0 \ldots N - 1$
- $\lambda(\theta)$ extremal for $\theta = 0, \pi$. For these values $\lambda = 1 \pm |\rho_S \pm \rho_A|$
- $\lambda > 0$ implies:

$$-1 \leq \rho_S + \rho_A \leq 1$$
$$-1 \leq \rho_S - \rho_A \leq 1$$

- If no serial correlations $\rho^* \in [-1, 1]$
- Instances when $\rho^* > 1$: evidence of serial correlations

- Impact of $\rho^* > 1$ on trading desk: price with the right realized volatilities, 100% correlation \rightarrow lose money !!
Example with basket option

- Sell 6-month basket option on basket of Japanese stock & French stock.
  
  Payoff is \( \left( \sqrt{\frac{S_1^T S_2^T}{S_0^T S_0^T}} - 1 \right)^+ \)

- Basket is lognormal with volatility given by \( \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2} \)

- Use following "historical" data:

  - Realized vols are 21.8% for \( S_1 \), 23.6% for \( S_2 \). Realized correlations are \( \rho_S = 63.3\% \), \( \rho_A = 57.6\% \): \( \rho^* = 121\% \).
• Backtest delta-hedging of option with:
  • implied vols = realized vols
  • different implied correlations

• Initial option price and final P&L:

![Graph showing correlation vs P&L](image)

- Final P&L vanishes when one prices and risk-manages option with an implied correlation $\rho \approx 125\%$. 
It is possible to price and risk-manage options on asynchronous securities using the standard synchronous framework, provided special correlation estimator is used.

Correlation estimator quantifies correlation that is materialized as cross-Gamma P&L.

Same technique applies to asynchronous FX / Swap rates / ...

Correlation swaps and options have to be priced with different correlations.

Serial correlations may push realized value of $\rho^*$ above 1: a short correlation position will lose money, even though one uses the right vols and 100% correlation.