What generates equity smiles?

Lorenzo Bergomi

lorenzo.bergomi@sgcib.com

Global Markets Quantitative Research

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Outline

▶ Sources of the equity smile?
▶ Historical distribution of daily returns
▶ An SV model with conditional non-Gaussian returns
▶ Impact on vanilla smiles
▶ Impact on path-dep payoffs
Intro – 1

- What generates equity smiles? Supply & demand
- OK, but what should the "fair" smile look like?

- If no vanillas exist - have to quote one
  - Delta-hedge vanilla ⇒ gamma/theta P&L
  - "Fair" ATMF skew given by covariance of spot & future realized variance

- If ATM vanillas liquid
  - Consider call spread position centered on ATMF such that $\Gamma = 0$
  - Initially $\Gamma = 0$, but $\Gamma + /\Gamma -$ when spot moves
  - Trade dynamically ATMF vanillas to cancel $\Gamma$
  - Carry P&L: spot/ATMF vol cross-gamma + ATMF vol gamma. Latter risk smaller

⇒ In liquid markets ATMF skew measures implied level of spot/ATMF vol covariance (not spot/future realized vol)

- At order 1 in vol-of-vol:

$$S_T = \left. \frac{d\hat{\sigma}_{KT}}{d\ln K} \right|_F = \frac{1}{2\hat{\sigma}_{T}^3 T} \int_0^T \frac{T-t}{T} \langle d\ln S_t \; d\hat{\sigma}_{T,t}^2 \rangle$$

$\hat{\sigma}_{T,t}$: ATMF/VS vol at $t$ for maturity $T$
Intro – 2

- So far considered P&L at order 2 in $\delta S, \delta \hat{\sigma}_T$

- What about large spot moves?
  - Responsible for steep smiles?

- Do large drawdowns generate a significant portion of the vanilla smile?

- Do they impact other derivatives?
Unconditional distribution of daily returns – 1

- Take 1 century worth of DJIA daily closes → daily returns \( r_i = \frac{S_i}{S_{i-1}} - 1 \)
- Separately normalize positive/negative returns:
  \[
  r_i \rightarrow \bar{r}_i = \frac{r_i}{\sqrt{\langle r_i^2 \rangle}}
  \]
- Rank negative returns from lowest to highest. Define empirical cumulative density of normalized negative returns as:
  \[
  P[\bar{r} \leq \bar{r}_i] = \frac{1}{2} \frac{i}{N^-}
  \]
- Graph \( \log_{10} P[\bar{r} \leq r] \). Do same for positive returns
- Compare with (a) lognormal distribution, (b) Student distribution:
  \[
  \rho_\mu (\bar{r}) = \frac{\Gamma \left( \frac{1+\mu}{2} \right)}{\sqrt{\mu \pi} \Gamma \left( \frac{\mu}{2} \right)} \frac{1}{\left( 1 + \frac{\bar{r}^2}{\mu} \right)^{\frac{1+\mu}{2}}} \propto \frac{1}{|\bar{r}|^{1+\mu}} \text{ for } \bar{r} \text{ large}
  \]

\( \mu \): number of degrees of freedom

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Unconditional distribution of daily returns – 2

- Dow Jones: 1900-2014

- 14103 negative returns. $\min(\bar{r}_i) = -19.6$. About 100 values of $\bar{r}_i \leq -4$.
- 15657 positive returns. $\max(\bar{r}_i) = 14.3$.
- Fit with $\mu = 3.2$

- Student distribution
  - The smaller $\mu$ the thicker the tails. Only moments of order $< \mu$ exist
  - Variance $= \frac{\mu}{\mu-2}$. Kurtosis $= \frac{6}{\mu-4}$.
  - For $\mu \to \infty$ converges to Gaussian distribution

$\Rightarrow$ Fit OK
Unconditional distribution of daily returns – 3

- Cumulative densities for different values of \( \mu \), all with \( E[\bar{r}^2] = 1 \).

\[ \mu \in [3, 4] \] acceptable

- Left/right tails of empirical density similar ?? Yes
  - Dow Jones daily returns: Sep–Dec 1929 / Sep–Dec 1987
Unconditional distribution of daily returns – 4

- Other example: HSCEI index, 1993-2014

Daily returns of equity indexes well captured by Student distribution
So far have looked at unconditional distribution – lumps together very different volatility regimes.

- Write

\[ r_i = \sigma_i \sqrt{\delta t} z_i \quad E[z_i^2] = 1 \]

- Fat tails of \( r_i \) due to randomness of \( \sigma_i \)?
- Look at conditional distribution
Conditional distribution of daily returns

\[ r_i = \sigma_i \sqrt{\delta t} \, z_i \]

- No easy access to \( \sigma_i \) – unless intraday data available.
  - Intrinsic noise of estimator of \( \sigma_i \) pollutes estimation of tails of \( z_i \)
- Proxy for \( \sigma_i \): 1-year realized vol
- Dow Jones: 1900-2014 again

\[ \mu \text{ larger than in unconditional distribution: OK} \]

⇒ Even accounting for randomness of volatility, daily returns are fat-tailed
⇒ \( z_i \) markedly non-Gaussian ⇒ sizeable 1-day conditional smile
⇒ How does the 1-day conditional smile impact derivatives?
SV model with conditional 1-day smile

- 1-day smile generates higher-order contributions to carry P&L, beyond gammas & cross-gammas of spot/implied vols

⇒ Need SV model to assess impact of (unhedgeable) 1-day smile risk
  - SV part of model sets $\sigma_i$, i.e. scale of daily returns
  - 1-day smile params govern 1-day conditional density
  - ... while keeping (a) vols of (implied) vols, (b) covariances of spot & (implied) vols unchanged.

- Start with 2-factor fwd variance workhorse

  - Dynamics of inst. fwd variances $\xi^T_t$:\(^2\)

  \[
  \frac{d\xi^T_t}{\xi^T_t} = (2\nu) \alpha \left( (1-\theta)e^{-k_1(T-t)}dW^X_t + \theta e^{-k_2(T-t)}dW^Y_t \right)
  \]

  - Curve $\xi^T_t$ a function of two OU processes $X_t, Y_t$:

  \[
  \xi^T_t = \xi^T_0 e^{(2\nu)\alpha \left[ (1-\theta)e^{-k_1(T-t)}X_t + \theta e^{-k_2(T-t)}Y_t \right] - \frac{4\nu^2\alpha^2}{2}}
  \]

  \[
  dX_t = -k_1 X_t dt + dW^X_t \quad dY_t = -k_2 Y_t dt + dW^Y_t
  \]

\(^2\alpha\) normalization factor: $\alpha = 1/\sqrt{(1-\theta)^2 + \theta^2 + 2\rho_{XY}\theta(1-\theta)}$ ⇒ $\text{vol}(\xi^T_t) = 2\nu$ ⇒ $\text{vol}(\sqrt{\xi^T_t}) = \nu$
SV model – 1

- Process for $S_t$:

$$dS_t = (r - q)S_t dt + \sqrt{\xi_t} S_t dW^S_t$$

$$\langle dW^S dW^X \rangle = \rho_{SX} dt \quad \langle dW^S dW^Y \rangle = \rho_{SY} dt$$

- Instantaneous vol of VS vol $\hat{\sigma}_T$ of maturity $T$ – for flat TS of VS vols:

$$\text{vol}(\hat{\sigma}_T) = \nu \alpha \sqrt{(1 - \theta)^2 l^2 (k_1 T) + \theta^2 l^2 (k_2 T) + 2 \rho_{XY} \theta (1 - \theta) l (k_1 T) l (k_2 T)}$$

$$l(x) = \frac{1 - e^{-x}}{x}$$

$$\text{vol}(\hat{\sigma}_{T=0}) = \nu$$

- Vol of ATMF vol $\approx$ vol of VS vol

- ATMF skew at order 1 in vol-of-vol for flat TS of VS vols:

$$S_T = \nu \alpha \left[ (1 - \theta) \rho_{SX} \frac{k_1 T - (1 - e^{-k_1 T})}{(k_1 T)^2} + \theta \rho_{SY} \frac{k_2 T - (1 - e^{-k_2 T})}{(k_2 T)^2} \right]$$
SV model – 2

- Params $\theta, k_1, k_2, \rho_{XY}$? $\Rightarrow$ so that $\text{vol}(\hat{\sigma}_T)$ matches power-law benchmark

$$\text{vol}(\hat{\sigma}_T)_{\text{Benchmark}} = \nu_0 \left( \frac{3m}{T} \right)^\alpha$$

over range $[T_{\text{min}}, T_{\text{max}}]$.

- Typically, $\alpha = 0.4$, $\nu_0 = 60\%$ (realized) / $100\%$ (implied)
  - Sets for $\alpha = 0.4$, $\nu_0 = 100\%$, range $[1\text{m}, 5\text{y}]$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>120.9%</th>
<th>135.8%</th>
<th>174.0%</th>
<th>178.2%</th>
<th>181.9%</th>
<th>185.1%</th>
<th>190.1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>57.9%</td>
<td>30.1%</td>
<td>24.5%</td>
<td>23.8%</td>
<td>23.4%</td>
<td>23.1%</td>
<td>22.8%</td>
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<tr>
<td>$k_1$</td>
<td>0.58</td>
<td>2.59</td>
<td>5.35</td>
<td>6.02</td>
<td>6.65</td>
<td>7.26</td>
<td>8.34</td>
</tr>
<tr>
<td>$k_2$</td>
<td>1.19</td>
<td>0.32</td>
<td>0.28</td>
<td>0.27</td>
<td>0.25</td>
<td>0.24</td>
<td>0.22</td>
</tr>
<tr>
<td>$\rho_{XY}$</td>
<td>-95%</td>
<td>-50%</td>
<td>0%</td>
<td>20%</td>
<td>40%</td>
<td>60%</td>
<td>99%</td>
</tr>
</tbody>
</table>

- No over-parametrization. Different sets $\Rightarrow$ different short vol/long vol correlations

- Spot/factor corrs $\rho_{SX}$, $\rho_{SY}$ such that:
  - Either generate given level & term-structure of covariances of spot & ATMF vols
  - Or generate desired term structure of ATMF skew. Typically:

$$S_T \propto \frac{1}{T^\gamma} \quad \text{with} \quad \gamma \in [0.3, 0.7], \quad \text{range} \quad [1\text{m}, 5\text{y}]$$
SV model with conditional 1-day smile – 1

» Set time scale $\Delta = 1$ day

» Fwd variances: simulate increments $\delta X, \delta Y$ of OU processes $X, Y$

\[
\delta X = \int_t^{t+\Delta} e^{-k_1(t+\Delta-u)} dW_u^X \quad \delta Y = \int_t^{t+\Delta} e^{-k_2(t+\Delta-u)} dW_u^Y
\]

» Spot increment:

\[
S_{t+\Delta} = S_t \left[ 1 + (r - q) \Delta + \sigma_t \delta Z \right]
\]

with

\[
\sigma_t = \sqrt{\frac{1}{\Delta} \int_t^{t+\Delta} \xi_t^\tau d\tau} \approx \sqrt{\xi_t^\tau} \text{ if } \Delta \text{ small}
\]

⇒ In standard 2F model

\[
S_{t+\Delta} = S_t \left[ 1 + (r - q) \Delta + \sigma_t \delta W^S \right]
\]

⇒ Here $\delta Z$ fat-tailed, no longer Gaussian
SV model with conditional 1-day smile – 2

- $\delta Z$: 2-sided Student distribution with params $\mu_+, \mu_-$

- Histo. positive & negative returns $\approx$ equally probable $\Rightarrow$ 1-day ATM digital $\approx \frac{1}{2}$

- In model, want ability to set 1-day ATM skew at will

$\Rightarrow$ Introduce $p^+, p^-$: probabilities of positive/negative returns

\[
\delta Z = \begin{cases} 
\sigma_+ \sqrt{\Delta} \mid X_{\mu_+} \mid & \text{with probability } p_+ \\
- \sigma_- \sqrt{\Delta} \mid X_{\mu_-} \mid & \text{with probability } p_-
\end{cases}
\]

$X_\mu$: Student random variable with $\mu$ degrees of freedom

$\Rightarrow$ $\sigma_+, \sigma_- \text{ such that } E[\delta Z] = 0, E[\delta Z^2] = \Delta$

- Need to correlate $\delta Z$ with $\delta X, \delta Y$
SV model with conditional 1-day smile – 3

Define function \( f \) that maps Brownian increment \( \delta W^S \) into \( \delta Z \):

\[
\frac{\delta Z}{\sqrt{\Delta}} = f \left( \frac{\delta W^S}{\sqrt{\Delta}} \right)
\]

\[
\begin{cases}
  x \leq \mathcal{N}_G^{-1}(p_-): & f(x) = \zeta_- \sqrt{\frac{\mu_- - 2}{\mu_-}} \mathcal{N}_\mu_- \left( \frac{\mathcal{N}_G(x)}{2p_-} \right) \\
  x \geq \mathcal{N}_G^{-1}(p_-): & f(x) = \zeta_+ \sqrt{\frac{\mu_+ - 2}{\mu_+}} \mathcal{N}_\mu_+ \left( \frac{1}{2} + \frac{\mathcal{N}_G(x) - p_-}{2p_+} \right)
\end{cases}
\]

\( \mathcal{N}_G \): CDF of standard normal variable, \( \mathcal{N}_G^{-1} \) its inverse

\( \mathcal{N}_\mu^{-1} \): inverse CDF of Student random variable with \( \mu \) degrees of freedom

\( \zeta^+, \zeta^- \) given by:

\[
\zeta_+ = \frac{p_- \alpha_-}{\sqrt{p_+ (p_- \alpha_-)^2 + p_- (p_+ \alpha_+)^2}} \\
\zeta_- = \frac{p_+ \alpha_+}{\sqrt{p_+ (p_- \alpha_-)^2 + p_- (p_+ \alpha_+)^2}}
\]

with \( \alpha_+ = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\mu_+ - 2}}{\mu_+ - 1} \frac{\Gamma \left( \frac{1 + \mu_+}{2} \right)}{\Gamma \left( \frac{\mu_+}{2} \right)} \) and likewise for \( \alpha_- \)

Mapping function \( f \) built once and for all.

\[
E[f(x)] = 0 \quad E[f^2(x)] = 1
\]
SV model with conditional 1-day smile – 4

▶ Now able to generate $\delta Z$ from Brownian increment $\delta W^S$: $\frac{\delta Z}{\sqrt{\Delta}} = f\left(\frac{\delta W^S}{\sqrt{\Delta}}\right)$

▶ Last part of job: correlate $\delta W^S$ with $\delta X, \delta Y$
  ▶ Covariances $E[\delta Z\delta X], E[\delta Z\delta Y]$ need to stay fixed

▶ In fat-tailed version of model, use correlations $\rho_{sx}^*, \rho_{sy}^*$ such that:

$$E^*\left[\delta Z\delta X\right] = E\left[\delta W^S\delta X\right] \quad \text{likewise for } E[\delta Z\delta Y]$$

▶ Standard 2F model: $\delta X = I(k_1\Delta)(\rho_{sx}\delta W^S + \ldots)$ $I(x) = \frac{1-e^{-x}}{x}$

▶ Fat-tailed 2F model: $\delta X = I(k_1\Delta)(\rho_{sx}^*\delta W^S + \ldots)$

▶ Equate covariance of $\delta X$ with $\delta Z, \delta W^S$:

$$E^*\left[\delta Z(\bullet \rho_{sx}^*\delta W^S)\right] = E\left[\delta W^S(\bullet \rho_{sx}\delta W^S)\right] = \bullet \rho_{sx}\Delta$$

▶ Yields:

$$\frac{\rho_{sx}^*}{\rho_{sx}} = \frac{\Delta}{E[\delta Z\delta W^S]} = \frac{1}{\int \frac{e^{-x^2/2}}{\sqrt{2\pi}} \times f(x)dx}$$

⇒ Rescaling of spot/vol correlations same for all factors:

$$\frac{\rho_{sy}^*}{\rho_{sy}} = \frac{\rho_{sx}^*}{\rho_{sx}} \geq 1 \quad \text{(Cauchy-Schwarz)}$$
SV model with conditional 1-day smile – 5

- Fait-tailed 2F model
  - Standard simulation of 2 OU processes $X, Y$ with corrs $\rho_{SX}^*, \rho_{SY}^*$ with $W^S$.
  - Spot simulation: no harder than in standard 2F model:
    \[
    S_{t+\Delta} = S_t \left[ (r-q)S_t \Delta + \sigma_t \sqrt{\Delta} f \left( \frac{\delta W^S}{\sqrt{\Delta}} \right) \right]
    \]
  - Pricing time similar to 2F standard model – in practice $\sigma_t = \sqrt{\xi_t}$
  - Can vary 1-day mile (i.e. $f$) while leaving dynamics of vols unchanged: vols of implied vols, corrs of spot & implied vols
    - 1-day smile params only change conditional density of normalized daily returns
    - Neither possible with jump/diffusion, nor with time-changed Lévy processes $L_{\tau_t}$
      - Conditional skewness & kurtosis fixed, correlation of spot and vols fixed
    - Continuous limit of model ?? Depends on scaling of $p^+ (\Delta) - \frac{1}{2}$ and $\mu (\Delta)$ as $\Delta \to 0$. 
1-day smile

- Left: $p_+ = p_- = \frac{1}{2}$. Right: $p_+ \neq p_-$. 

\[ \mu_+ = \mu_-, \sigma_t = 20\%. \]

Smile is obtained by numerical integration

$p_+, \mu_+, \mu_-$ do what they’re supposed to do.
Vanilla smile – 1

- Parameters of 2F model (typical of STOXX50 – July 2014)

<table>
<thead>
<tr>
<th>ν</th>
<th>θ</th>
<th>k₁</th>
<th>k₂</th>
<th>ρ_XY</th>
<th>ρ_SX</th>
<th>ρ_SY</th>
</tr>
</thead>
<tbody>
<tr>
<td>257%</td>
<td>15.1%</td>
<td>8.96</td>
<td>0.46</td>
<td>40%</td>
<td>-74.6%</td>
<td>-13.7%</td>
</tr>
</tbody>
</table>

- 3m and 1y smiles for different \( \mu_+ = \mu_- \), \( p_+ = 0.5 \), VS vols flat at 20%.

Std: standard 2F model – equivalent to \( \mu = \infty \)

- Std 2F model diffusive: algos for quasi-real-time vanilla smile generation – see book
- Fat-tailed 2F model is not: really have to price (delta-hedged) call/put payoffs

⇒ 1-day smile has minute impact on vanilla near-ATM smile
⇒ 1-day smile impacts tails – mostly OTM calls (for equities)
Vanilla smile – 2

- Impact of 1-day ATM digit. $\mu_+ = \mu_- = 4$.

![Graph showing impact of 1-day ATM digit](image)

- Scaling of 1-day skew contribution to 95/105 skew. Turn off stoch vol: $\nu = 0$

![Graph showing scaling of 1-day skew](image)

$\Rightarrow$ Contribution of 1-day skew to vanilla ATM skew $\propto \frac{1}{T}$: OK
Example 1: Daily cliquets – Gap notes – Crash puts

- Similar to CDS contract. Maturities 3m, 6m, 1y
- Receive Put (90%, 80%, 75%) or Put spread (90%/80%, 85%/75%) payoff on daily index returns
- Pay quarterly spread, starting at inception. Expires when 1st Put/Put spread is triggered
  - No delta, some vega – almost pure 1-day smile payoff
- Left: 1-day smile for different values of $\mu_-$. $\mu_+ = 4$, $p_+ = 0.5$, vol = 20%
- Right: upfront prices for 1-year 80% Crash Put – in basis points

Market prices very conservative, correspond to implied value of $\mu_- \approx 2.2$
Example 2: Var swaps

▶ \( \ln(\frac{S_{i+1}}{S_i})^2 \): VS other instance of daily cliquet

▶ Assume no dividends. Consider position: short VS/long vanilla replication of 
\(-2 \ln S\), delta-hedged

▶ Carry P&L cancels up to order 2 in \( \delta S \).

▶ Contribution of higher orders \( \Rightarrow \hat{\sigma}_{VS} \neq \hat{\sigma}_{Logswap} \)

▶ \((\hat{\sigma}_{VS} - \hat{\sigma}_{Logswap})\) for 1-year maturity, with/without stoch vol, for \( p_+ = \frac{1}{2} \).

\[ \begin{array}{c|cccc} \mu & \infty & 6 & 4 & 3 \\ \hline \nu = 0 & 0\% & 0\% & 0.02\% & 0.16\% \\
\nu = 257\% & 0.02\% & 0.04\% & 0.10\% & 0.29\% \\
\end{array} \]

\[ \begin{array}{c|cccccc} p_+ & 30\% & 40\% & 50\% & 60\% & 70\% \\ \hline \mu = 4, \nu = 257\% & -0.11\% & 0\% & 0.10\% & 0.23\% & 0.40\% \\
\end{array} \]

⇒ \( \mu = 4, p_+ = \frac{1}{2} \): relative mismatch \( \frac{\hat{\sigma}_{VS}}{\hat{\sigma}_{Logswap}} - 1 \) is \( \approx 0.10\%/20\% = 0.5\% \)

⇒ Direct backtesting on index returns? Slightly lower estimate:

\[ \frac{1}{2} \left( \frac{\langle r^2 \rangle}{\langle 2(e^r - 1) - 2r \rangle} - 1 \right) \quad r \text{ daily log-return} \]

⇒ Conclusion: \( \hat{\sigma}_{VS} - \hat{\sigma}_{Logswap} \): small impact of 1-day smile

⇒ Mostly impacted by dividend model
Conclusion

▶ SV model with handle on 1-day smile
  ... while keeping break-even levels of vommas & vannas unchanged

\[ r_i = \sigma_i \sqrt{\delta t} z_i \]

▶ Fwd variance model: sets scale \( \sigma_i \) of daily returns
▶ Additional parameters govern 1-day smile: \( \mu_+, \mu_-, p_+ \)
▶ Simulation no harder than in std 2F model

▶ Allows assessment of 1-day smile risk on derivatives
▶ Unhedgeable risk we're carrying: needs to be priced conservatively

ؤول Near-ATMF smile overwhelmingly generated by covariance of spot and ATMF/VS vols

\[ \frac{d\hat{\sigma}_{KT}}{d \ln K} \bigg|_F = \frac{1}{2\hat{\sigma}^3_T T} \int_0^T \frac{T - t}{T} \langle d\ln S_t d\hat{\sigma}^2_{T,t} \rangle \]

ؤول 1-day smile impacts tails of vanilla smile – mostly OTM calls

ؤول Larger impact on path-dep payoffs referencing daily returns
  ▶ Daily cliquets
  ▶ Var swaps
  ▶ Capped VSs, absswaps ...

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