Theta (and other Greeks): smooth barriers

Lorenzo Bergomi

lorenzo.bergomi@sgcib.com

Global Markets Quantitative Research

SOCIETE GENERALE

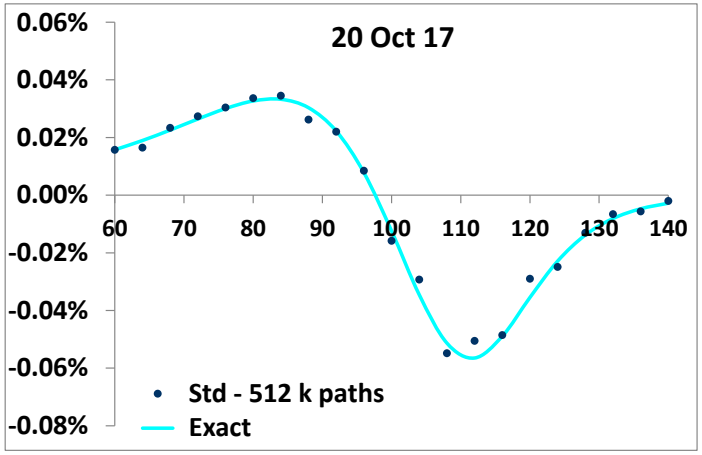
Lisbon, May 2018

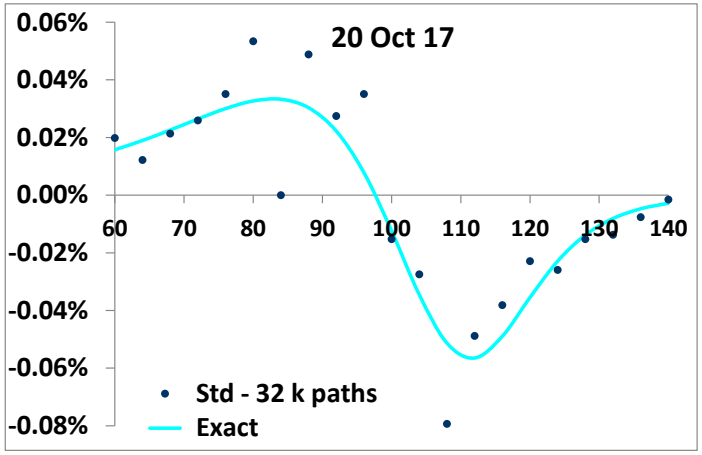
Outline

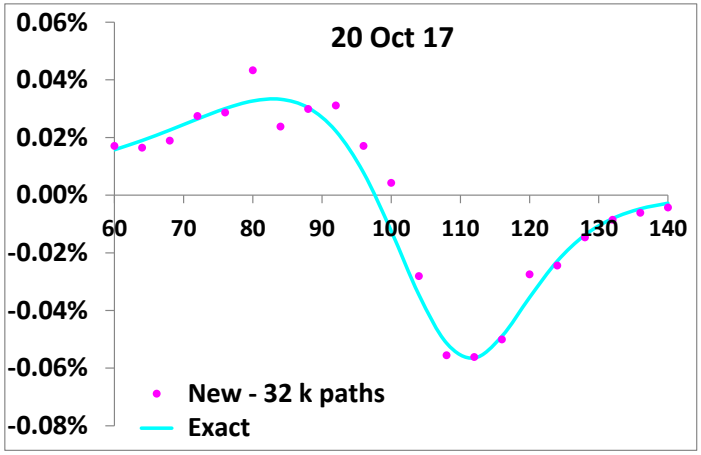
- Test of new Theta technique on digitals
- Smoothing digitals
- Deltas & Gammas of digitals
- Greeks of autocalls
- Determining gaps of recall barriers of autocalls

Try new Theta technique (cf Talk 1) on European digitals

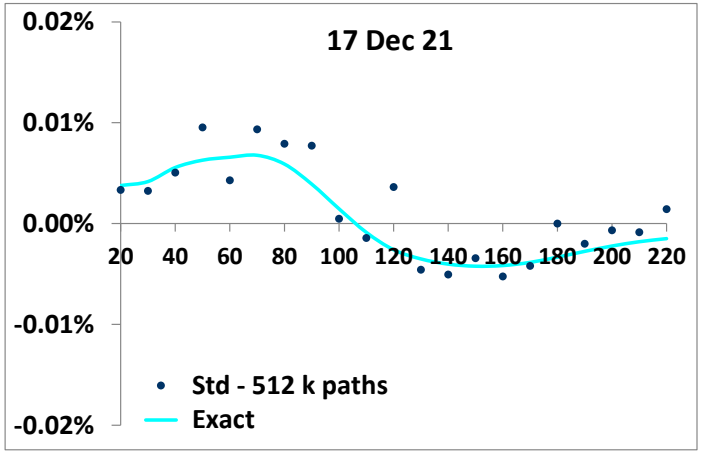
- SX5E smile, no repo, no divs. Pricing date = 21 Feb 2017.
  - Standard technique $P(t)$, $P(t+1 \text{ day})$ with 32k, 512k paths
  - New method $\delta t = 5$ tdays.
- Annualized Theta of digitals – strikes $[60, 140]$, mat = 20 Oct 17 – 6 months

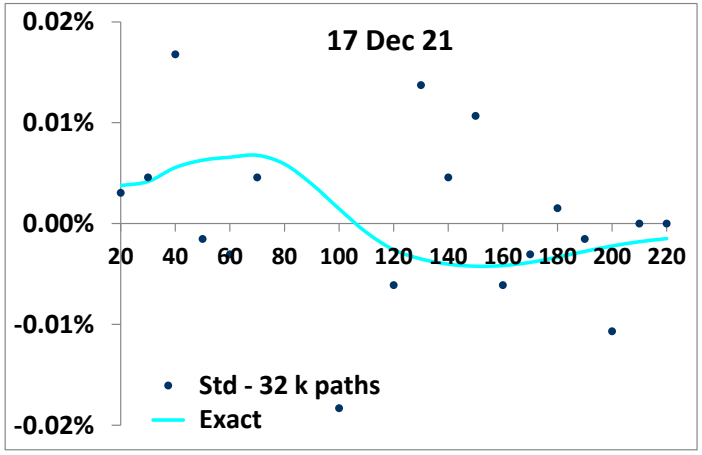


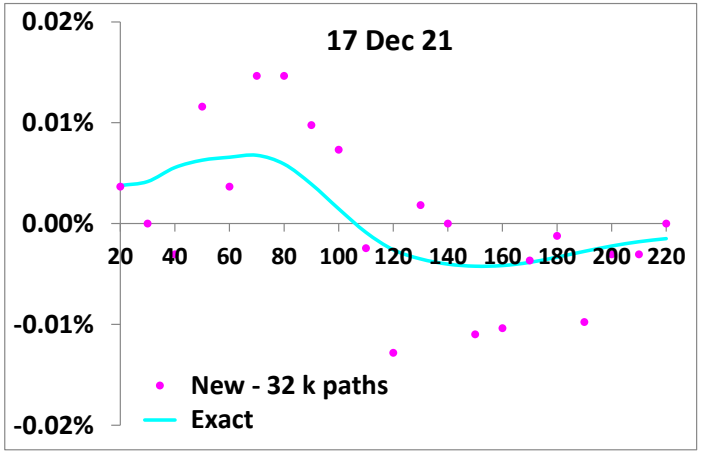




- Annualized Theta of digitals – strikes $[20, 220]$, mat = 17 Dec 21 – 4 years







⇨ New method not good enough – greeks of digitals too noisy ⇨ do something

Smoothing barriers – 1

- Start with digital option that pays $f(S)$ if $S > L$, 0 otherwise

$$f(S)\theta(S-L) \;=\; f(S)\theta(x) \qquad \text{with } x = 1 - \frac{L}{S}$$

$$\theta(x) \;=\; \mathbf{1}_{x \geq 0}$$

- Replace $\theta(x)$ with smooth version $\theta_\delta(x)$, smoothed over typical width δ, using kernel φ_δ:

$$\theta(x) \;\rightarrow\; \theta_\delta(x) \;=\; \int \varphi_\delta\,(u)\,\theta\,(x-u)\,du$$

- φ_δ given by (a) shape function h, (b) smoothing parameter δ:

$$\varphi_\delta(u) \;=\; \frac{1}{\delta} h\left(\frac{u}{\delta}\right)$$

- Smoothed barrier $\theta_\delta(x)$:

$$\theta_\delta(x) \;=\; H\left(\frac{x}{\delta}\right) \quad \text{with} \quad H\,(w) \;=\; \int_{-\infty}^{w} h\,(z)\,dz$$

Smoothing barriers – 2

- $P_\delta(L)$: price of barrier option with smoothed barrier θ_δ

$$\begin{aligned} P_\delta(L) &= E\left[f(S)\,\theta_\delta\Big(1-\frac{L}{S}\Big)\right] = E\left[f(S)\int\varphi_\delta(u)\,\theta\Big(1-\frac{L}{S}-u\Big)du\right] \\ &= \int\varphi_\delta(u)\,E\left[f(S)\,\theta\Big(1-\frac{L}{S(1-u)}\Big)\right]du = \int\varphi_\delta(u)\,P\Big(\frac{L}{1-u}\Big)du \\ &= \int h(z)\,P\Big(\frac{L}{1-\delta z}\Big)dz \end{aligned}$$

where $P(L)$ is price of standard barrier option: $P(L) = E\left[f(S)\,\theta\Big(1-\frac{L}{S}\Big)\right]$

- Expand in δ:

$$\begin{aligned} P_\delta(L) &= \int h(z)\,\Big((P(L) + a_1(L)\delta z + \frac{1}{2}a_2(L)\delta^2 z^2 + \ldots\Big)dz \\ &= P(L)\int h(z)dz + a_1(L)\,\overline{z}\cdot\delta + \frac{1}{2}a_2(L)\,\overline{z^2}\cdot\delta^2 + \ldots \end{aligned}$$

with $\overline{z^k} = \int h(z)z^k dz$

$a_1(L) = LP'$, $a_2(L) = 2LP'+L^2P''$, $a_3(L) = 6LP'+6L^2P''+L^3P'''$, $a_4(L) = 24LP'+36L^2P''+12L^3P'''+L^4P''''$

Smoothing barriers – 3

- No order-zero bias ⇨ $\int h(z)dz = 1$ (h density – if positive)
- No order-one bias ⇨ $\overline{z} = \int zh(z)dz = 0$
- Error starts at order 2: $\delta^2 \overline{z^2} = \delta^2 \int h(z)z^2 dz > 0$

⇨ Remove 2nd order error by using 2 different smoothings:

$$P_\delta^R = \frac{1}{3}(4P_\delta - P_{2\delta})$$

- No need to run 2 MCs. Run 1 MC with $\theta(x)$ replaced with $\theta_\delta^R(x)$ given by:

$$\theta_\delta^R = \frac{1}{3}(4\theta_\delta - \theta_{2\delta})$$

- If h symmetric get order-3 accuracy for free, otherwise use:

$$\theta_\delta^{R2} = \frac{1}{21}(32\theta_\delta - 12\theta_{2\delta} + \theta_{4\delta})$$

- If h symmetric:

$$P_\delta^R = P - \frac{1}{6}a_4\overline{z^4}\delta^4 + \ldots$$

$$P_\delta^{R2} = P + \frac{4}{21}a_4\overline{z^4}\delta^4 + \ldots$$

Smoothing barriers – 4

- Check out different kernels (shape functions)

$$\theta(x) \quad \rightarrow \quad \theta_\delta(x) = \int \varphi_\delta(u)\theta(x-u)du \;=\; H\left(\frac{x}{\delta}\right)$$

- Hat

z	$]-\infty,-1]$	$[-1,0]$	$[0,1]$	$[1,\infty[$
h	0	$1+z$	$1-z$	0
H	0	$\frac{1}{2}(1+z)^2$	$1-\frac{1}{2}(1-z)^2$	1

$$\overline{z^2} \;=\; \tfrac{1}{6} \qquad \overline{z^4} \;=\; \tfrac{1}{15}$$

- Gaussian

$$h(z) \;=\; \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad H(z) \;=\; \mathcal{N}(z)$$

$$\overline{z^2} \;=\; 1 \qquad \overline{z^4} \;=\; 3$$

⇨ $\text{Hat}(\delta) \;\approx\; \text{Gaussian}\left(\frac{\delta}{\sqrt{6}}\right)$

Smoothing barriers – 5

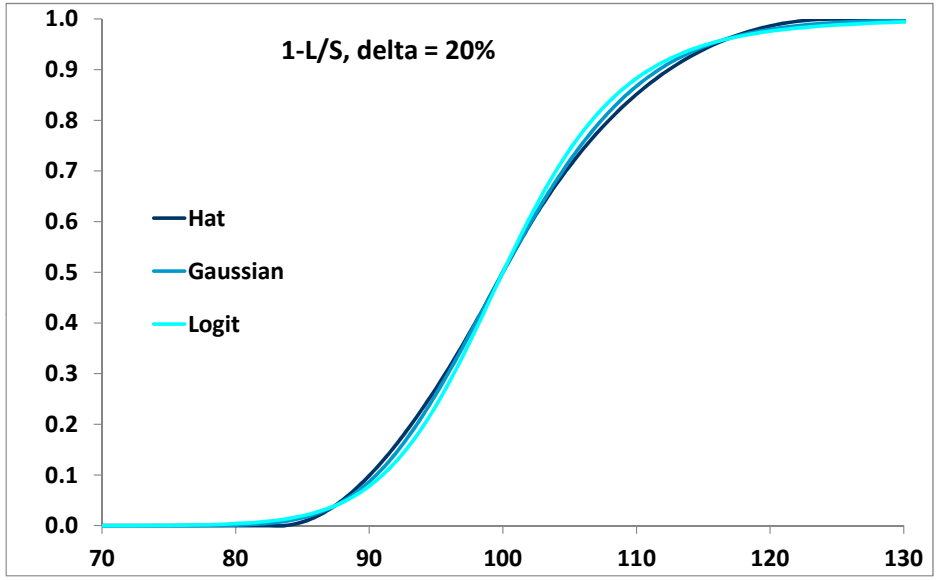
- Logit

$$h(z) = \frac{e^{-z}}{(e^{-z}+1)^2} \qquad H(z) = \frac{1}{1+e^{-z}}$$

$$\overline{z^2} = \frac{\pi^2}{3} \qquad \overline{z^4} = \frac{7\pi^4}{15}$$

⇨ $\text{Hat}(\delta) \approx \text{Logit}\left(\frac{\delta}{\pi\sqrt{2}}\right)$

- In practice kernels are similar. Compare $\text{Hat}(\delta)$, $\text{Gaussian}(\frac{\delta}{\sqrt{6}})$, $\text{Logit}(\frac{\delta}{\pi\sqrt{2}})$:



- Same kernels applied to $x = \ln \frac{S}{L}$ rather than $x = 1 - \frac{L}{S}$ ⇨ similar as well

Smoothing barriers – 6

- What if multiple barriers (Autocall) ? Payoff can always be written as:

$$f(\mathbf{S}) \prod_i \theta(S_i - L_i)$$

⇨ Use smoothed version of digits:

$$f(\mathbf{S}) \prod_i \theta_{\delta_i}\Big(1 - \frac{L_i}{S_i}\Big) \;=\; f(\mathbf{S}) \int \prod_i \varphi_{\delta_i}(u_i)\; \theta\Big(1 - \frac{L_i}{S_i} - u_i\Big) du_i$$

- Take expectation – smoothed price given by:

$$P_\delta(\mathbf{L}) \;=\; \int \prod_i \varphi_{\delta_i}(u_i) du_i \; P\Big(\frac{\mathbf{L}}{1 - \mathbf{u}}\Big)$$

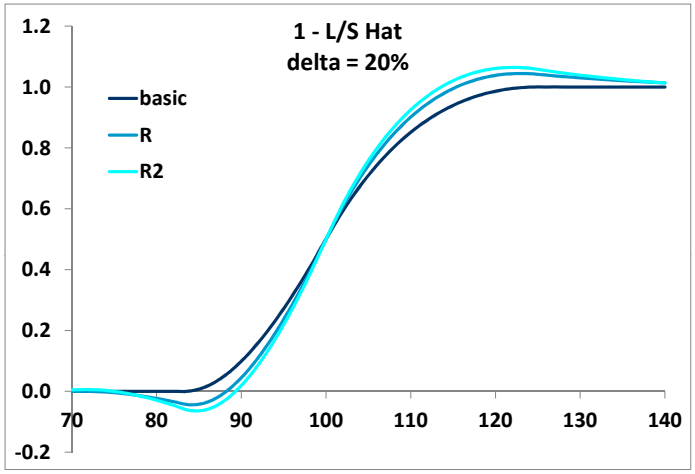
- Expand in powers of u_i. At 2nd order get usual diagonal terms + cross-terms:

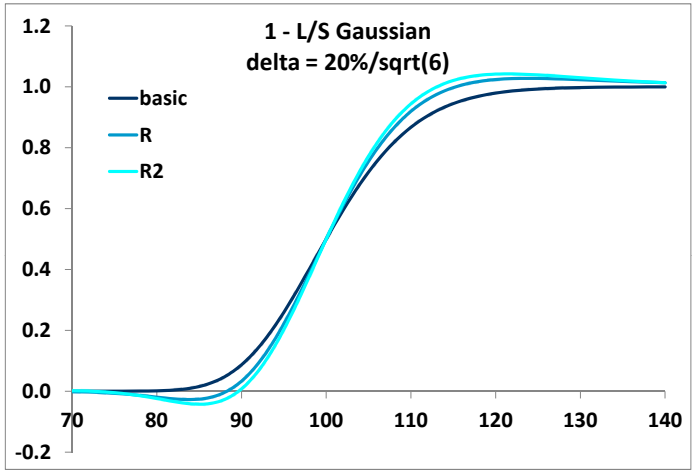
$$L_i L_j \frac{d^2 P}{dL_i dL_j} \int \varphi_{\delta_i}(u_i) u_i du_i \int \varphi_{\delta_j}(u_j) u_j du_j \;=\; L_i L_j \frac{d^2 P}{dL_i dL_j} \overline{u_i}\; \overline{u_j} \;=\; 0, \text{ if } h \text{ symmetric}$$

- Probabilistic interpretation of smoothing algo
  - Barrier L_i ⇨ Random barrier $\frac{L_i}{1 - \delta_i X_i}$
  - Random variables X_i independent and distributed with density h
  - To achieve 2nd order accuracy, run 2 MCs with δ, 2δ.

Smoothing barriers – bias – 1

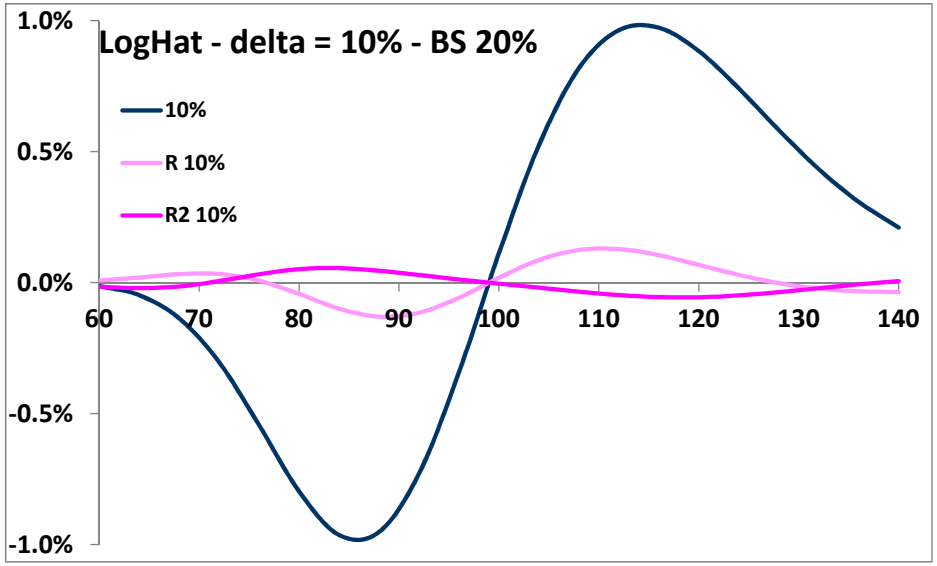
- Examples with $\delta = 20\%$: Hat (left) and Gaussian (right) kernels very similar





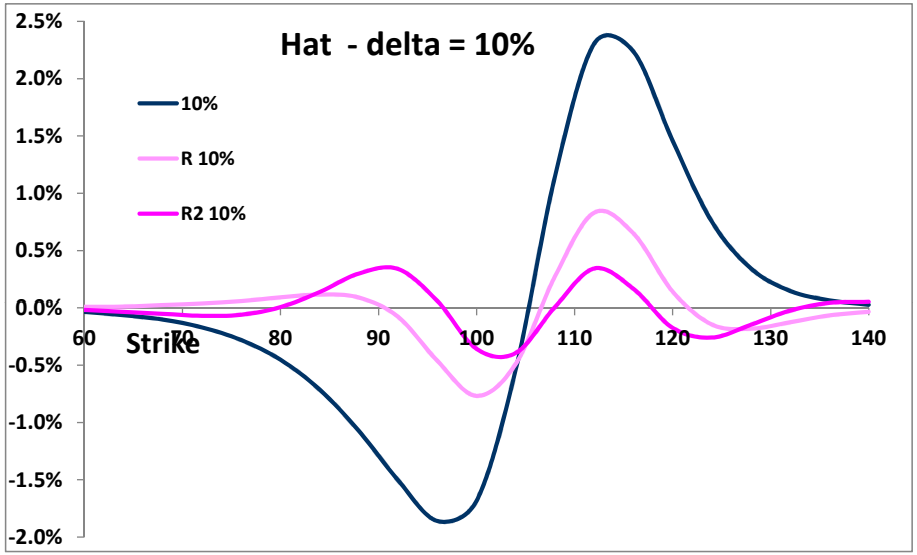
- Black-Scholes, $\sigma = 20\%$, LogHat kernel with $\delta = 10\%$

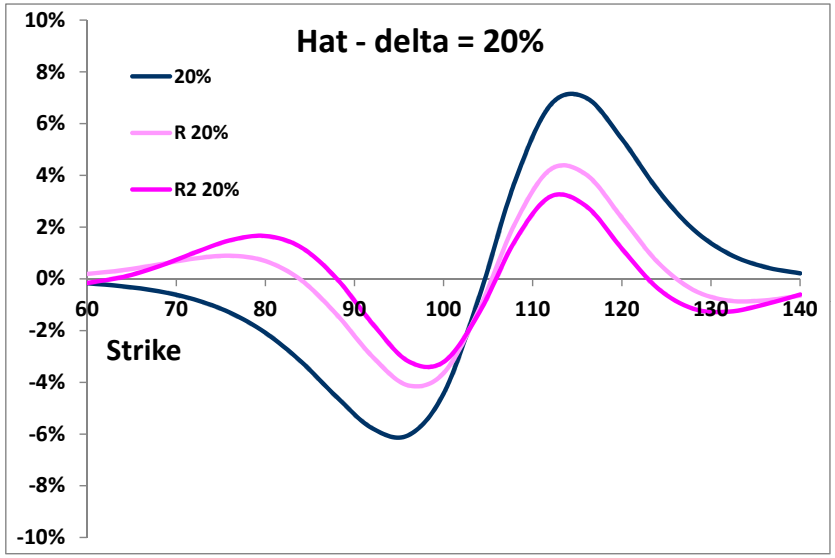
  $S = 100$, $T = 6$ months. Difference $(P - P_\delta)$ for digitals struck over $[60\%, 140\%]$



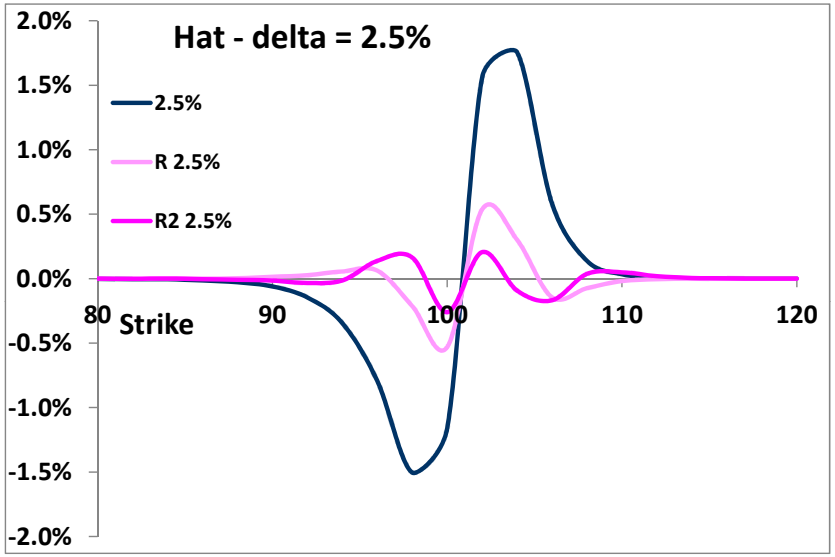
Smoothing digitals – bias – 2

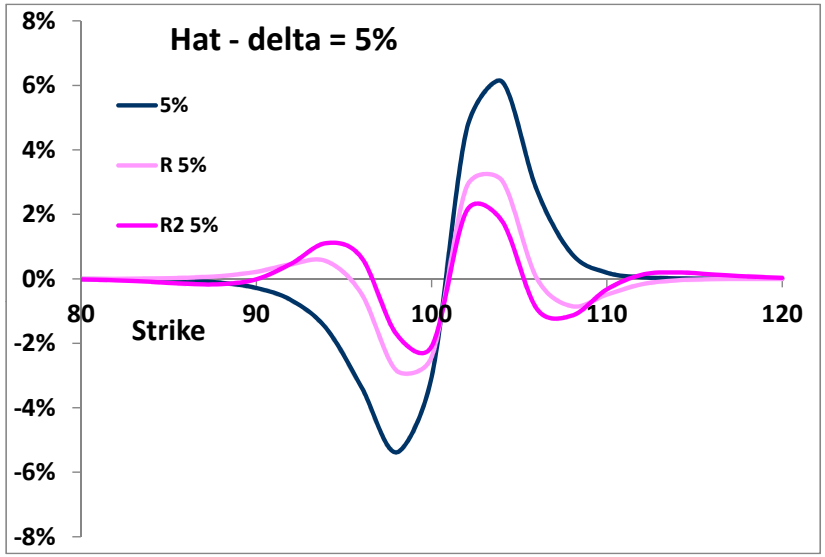
- $T = 6$m (SX5E, 21 Feb 17): errors wrt "exact prices" (512k paths). $\sigma_{ATM} = 17\%$





- $T = 1$m: $\sigma_{ATM} = 12\%$

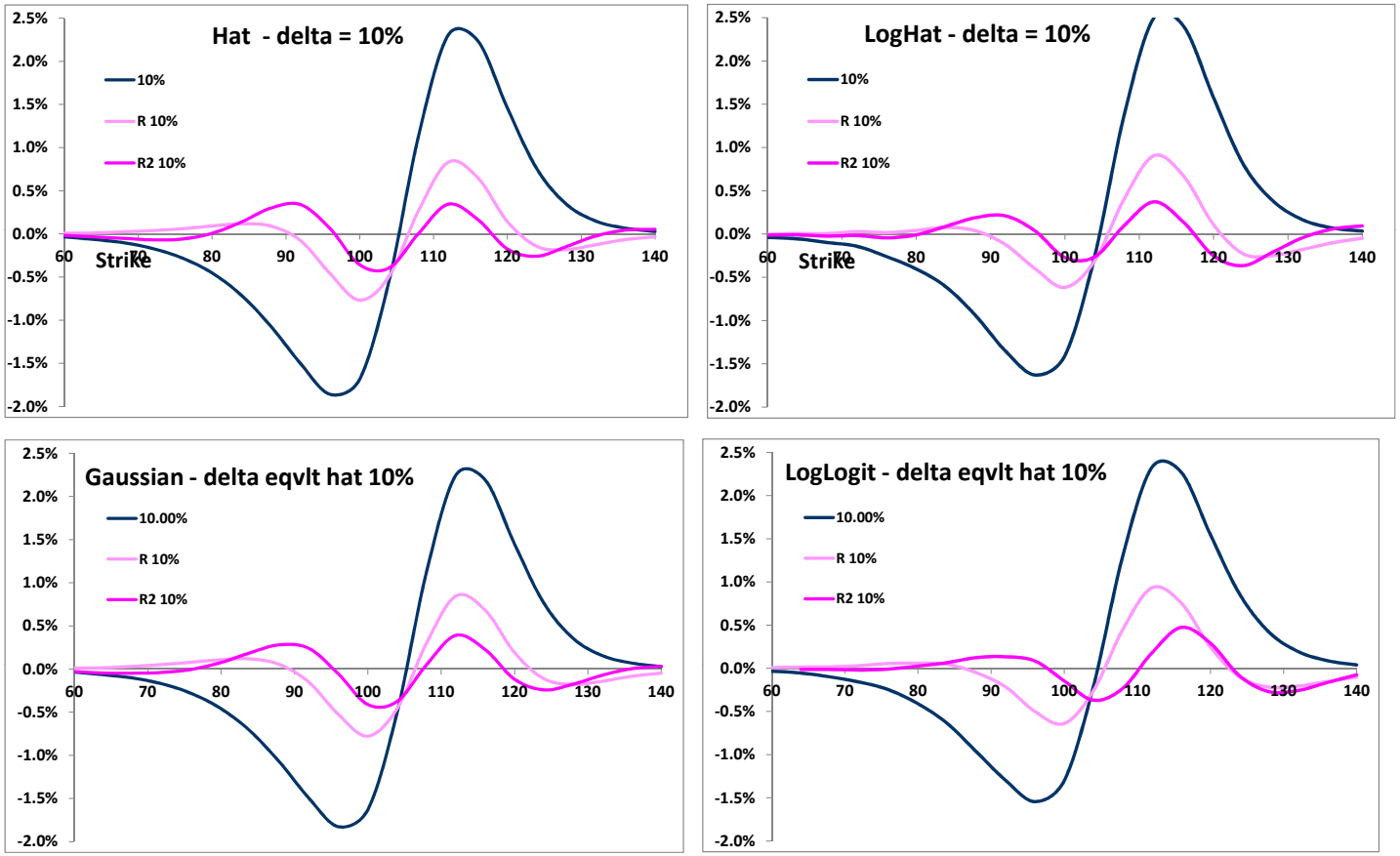




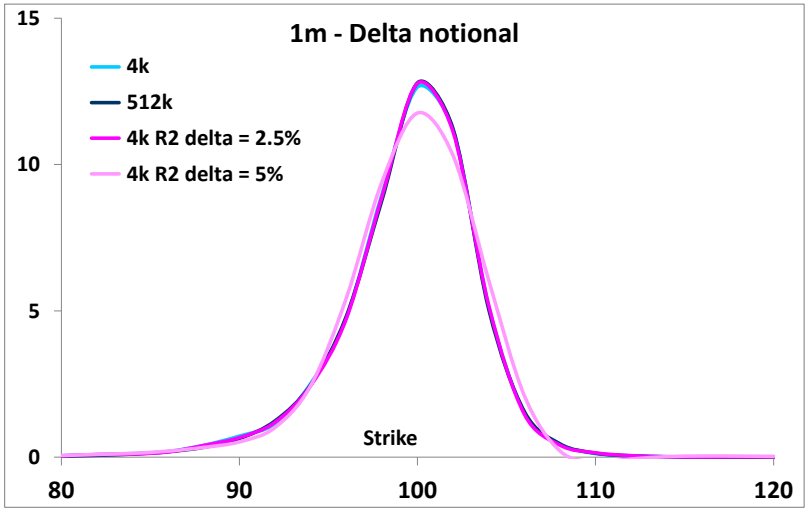
⇨ R2 somewhat better than R. How to pick δ ?

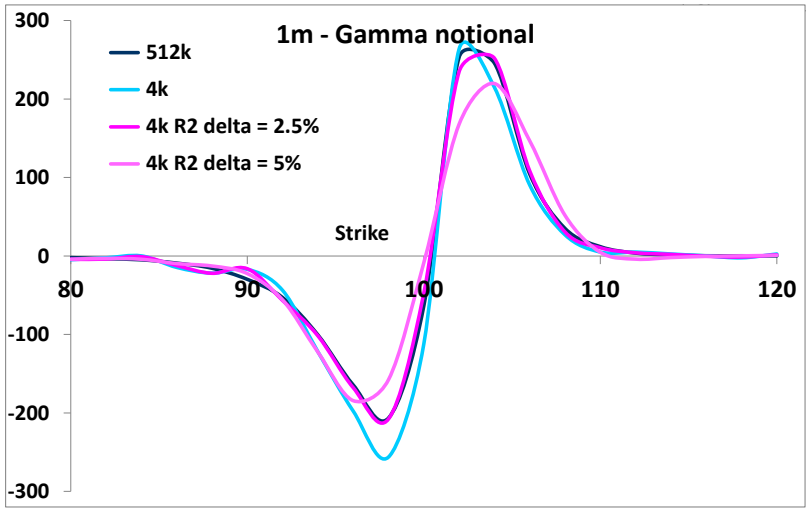
Smoothing digitals – bias – 3

- Smoothing (or randomizing) barrier ⇨ additional contribution to volatility
- In 6m/$\delta = 10\%$ and 1m/$\delta = 2.5\%$ examples, additional variance: $\frac{\delta_T^2}{6\widehat{\sigma}_T^2 T} \approx 10\%$
- Different kernels yield similar biases ⇨ in what follows use hat kernel



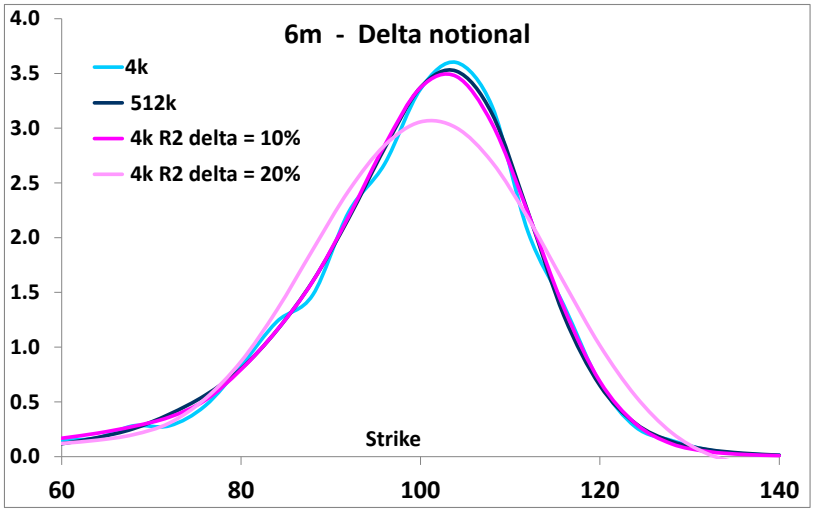
Smoothing digitals – Δ, Γ

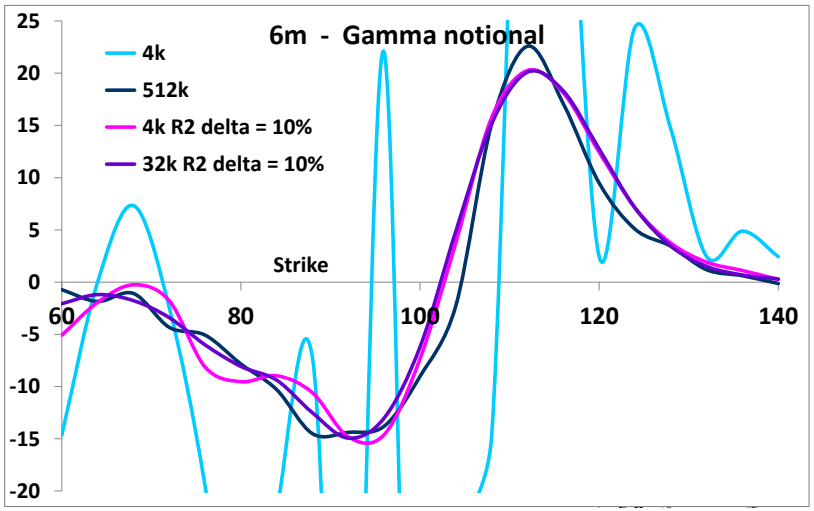




- 4k $\equiv$ 4k MC paths

⇨ $\delta = 5\%$: too large

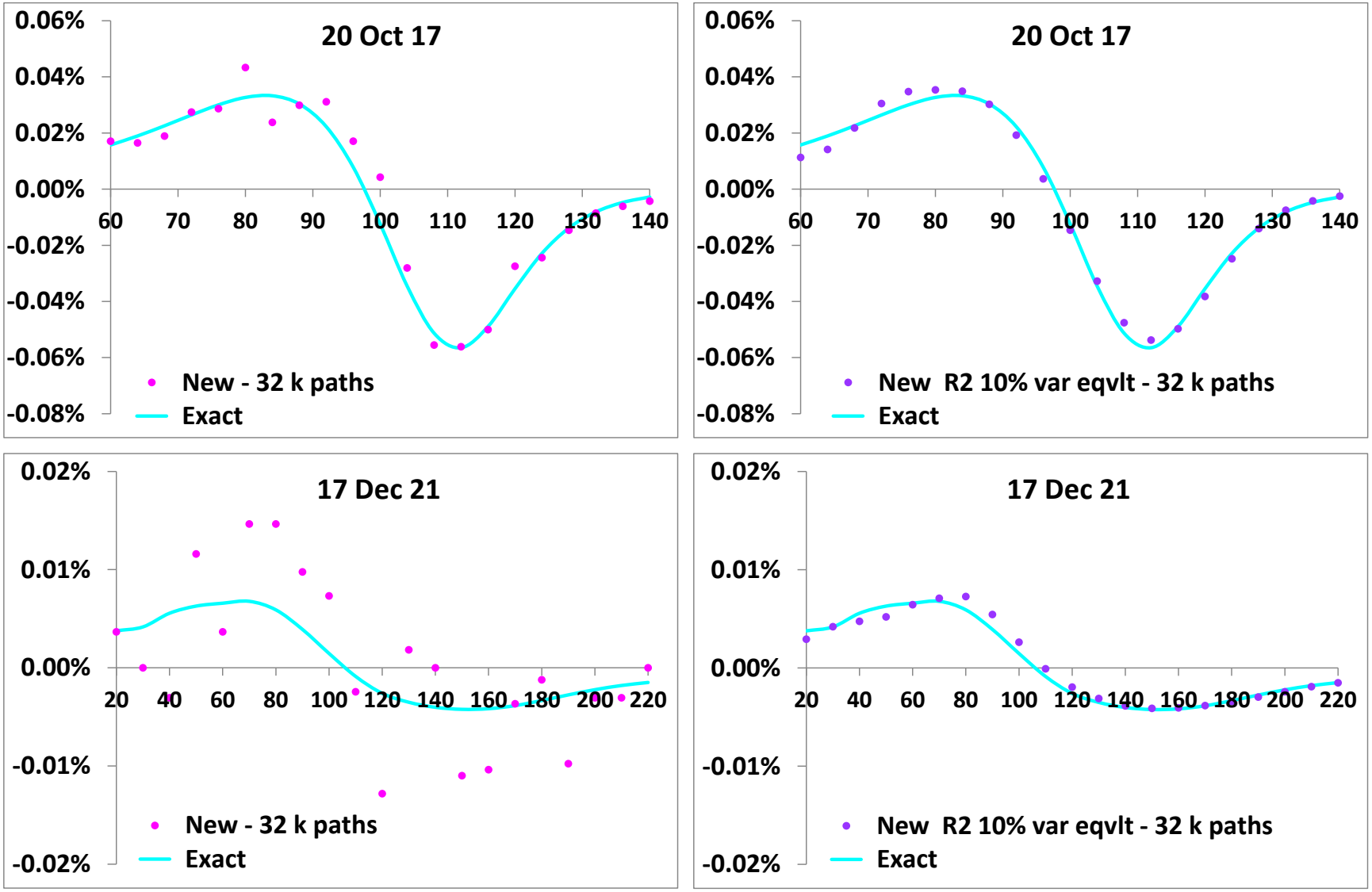




⇨ Particularly efficient for longer maturities

Smoothing barriers – Θ – 1

- Compare New, New with smoothing – 32k paths.
- Take $\delta t = 10$ days (1st maturity), $\epsilon = 50\%$. Hat kernel. $\delta_T = \sqrt{10\% \cdot 6\widehat{\sigma}_T^2 T}$



- OK, works. Now move on to autocalls

Autocalls – 1

- 3y autocall on Worst-of of Eurostoxx50/SP500/NIKKEI
- Recall every 6 months. Recall barriers L_i: 90, 90, 85, 80, 75, 60
- If recalled at T_i get $(1 + c_i)$ and autocall terminates. Coupon levels c_i: 3%, 6%, 9%, 12%, 15%, 18%
- At maturity get coupon & principal minus DI ATM put with DI barrier $L_{DI} = 60\%$
- DI barrier sampled at dates t_l for simplicity
- Market smiles as of 21 Feb 2017 – no rates/no repos

- Pricing with smoothed barriers
  - No more *if then else* clauses in scripts
  - Rewrite payoffs using θ functions, then replace with smoothed version ϕ
  - Introduce (a) survival "probability" p_i at T_i, (b) DI trigger "probability" p_{DI}
  - One underlying for simplicity:

$$p_i = \prod_{j<i} \left(1 - \theta_{\delta_j}\left(1 - \frac{L_j}{S_j}\right)\right)$$

$$p_{DI} = p_n\left(1 - \prod_l \theta_{\delta_l}\left(1 - \frac{L_{DI}}{S_l}\right)\right)$$

  - Payoff at recall dates T_i, including maturity T_n: $(1 + c_i)p_i$

    $+$ extra payoff at T_n if $S_n < L_n$: $p_{n+1} \cdot 1 - p_{DI}\left(1 - S_n\right)^+$

Autocalls – 2

- Autocall is written on Worst-Of
  - Solution 1: evaluate smoothed barrier on Worst-of:

$$\theta\Big(1 - \frac{L}{\min_k S^k}\Big) \quad \Rightarrow \quad \theta_\delta\Big(1 - \frac{L}{\min_k S^k}\Big)$$

  ⇨ Works but does not allow for individual setting of smoothing parameter δ

  - Solution 2: smooth barriers individually:

$$\theta\Big(1 - \frac{L}{\min_k S^k}\Big) = \prod_k \theta\Big(1 - \frac{L}{S^k}\Big) \quad \Rightarrow \quad \prod_k \theta_{\delta_k}\Big(1 - \frac{L}{S^k}\Big)$$

  ⇨ Individually adjust δ to each underlying's vol level

- In numerical tests use Solution 1, hat kernel, $\delta_T = \sqrt{10\% \cdot 6\widehat{\sigma}_T^2 T}$. $\widehat{\sigma}_T \approx 18\%$.
- Look at individual coupons $+ +$ split final payout into coupon $+$ 100 - DI Put.

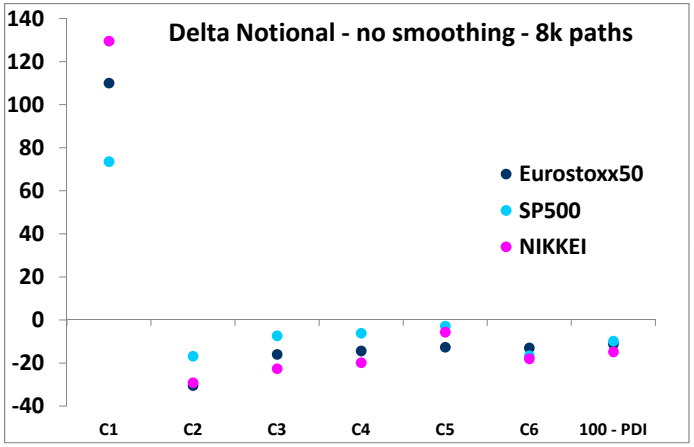
Autocalls – tests – Delta/Gamma

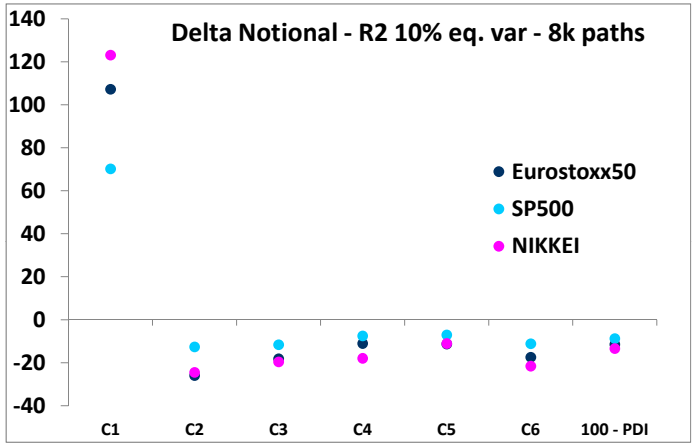
- Individual coupon prices: std vs smoothed barriers

No smoothing	Coupon 1	Coupon 2	Coupon 3	Coupon 4	Coupon 5	Coupon 6	100 - PDI	Total
8k	71.30	8.46	5.07	3.39	2.50	4.23	3.29	**98.25**
512k	71.13	8.39	5.40	3.53	2.49	4.18	3.27	**98.39**

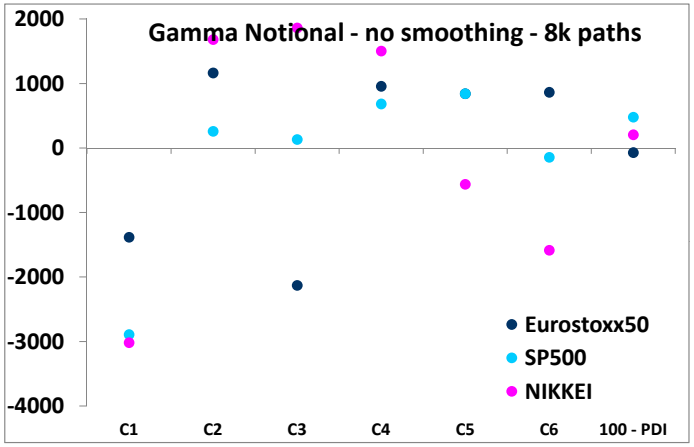
R2 10% eq. var	Coupon 1	Coupon 2	Coupon 3	Coupon 4	Coupon 5	Coupon 6	100 - PDI	Total
8k	71.44	8.19	5.21	3.53	2.55	4.04	3.31	**98.28**
512k	71.42	8.01	5.41	3.57	2.53	4.15	3.32	**98.41**

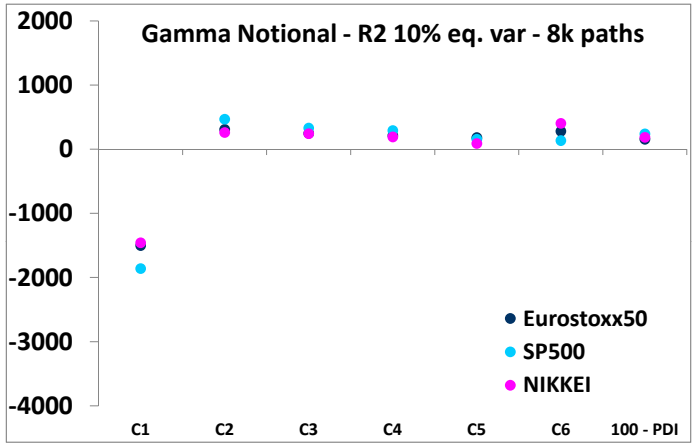
- Delta notionals $S_i \frac{dP}{dS_i}$ – 8k paths:





- Diagonal gamma notionals $S_i^2 \frac{d^2P}{dS_i^2}$ – 8k paths:





Autocalls – tests – Theta

- Annualized theta
  - Std: price at t, then at t+2 days: 8k, 32k, 512k, 2M paths
  - New: both prices at t + smoothing. $\delta t = 7$d, $\epsilon = 50\%$, no artificial maturity inserted

$$\Theta = \frac{P\left(\widehat{\sigma}_k^2\right) - P\left(\widehat{\sigma}_k^2 - \epsilon \frac{\delta t}{T_k} \widehat{\sigma}_k^2\right)}{\epsilon \delta t}$$

	Standard				New - smoothed		
	8k	**32k**	**512k**	**2M**	**8k**	**32k**	**512k**
Coupon 1	**39.0**	**64.8**	**39.0**	**28.3**	**25.1**	**24.5**	**24.1**
Coupon 2	**-23.6**	**-60.8**	**-9.1**	**-4.2**	**-3.0**	**-1.2**	**-0.5**
Coupon 3	**-12.1**	**10.3**	**-3.3**	**-2.2**	**-2.0**	**-2.9**	**-2.6**
Coupon 4	**22.5**	**-3.7**	**-3.4**	**0.0**	**-1.2**	**-0.4**	**-2.4**
Coupon 5	**2.6**	**3.2**	**-2.0**	**-4.5**	**-3.4**	**-3.2**	**-2.3**
Coupon 6	**-34.2**	**0.7**	**-5.6**	**-5.1**	**-5.0**	**-4.7**	**-4.3**
100 - PDI	**9.9**	**-4.3**	**-5.7**	**-4.3**	**-3.4**	**-4.2**	**-4.1**
Total	**4.0**	**10.2**	**9.8**	**8.0**	**7.2**	**7.9**	**7.9**

- 1-day theta is indeed small quantity. $8/365 \approx 2$bps/day.

⇨ New Theta + smoothing performs OK

Autocalls – tests – Theta – 2

- Is the efficiency coming from the new Theta technique or from barrier smoothing?
- Compute Theta with std technique, but with smoothed barriers

	Standard			Standard - smoothed		
	8k	**32k**	**512k**	**8k**	**32k**	**512k**
Coupon 1	39.0	64.8	39.0	45.0	55.5	28.7
Coupon 2	-23.6	-60.8	-9.1	-14.1	-30.1	-2.4
Coupon 3	-12.1	10.3	-3.3	-4.2	-2.7	-1.7
Coupon 4	22.5	-3.7	-3.4	-11.3	-6.6	-2.1
Coupon 5	2.6	3.2	-2.0	4.8	0.2	-2.8
Coupon 6	-34.2	0.7	-5.6	-4.0	-3.8	-6.0
100 - PDI	9.9	-4.3	-5.7	-3.4	-4.1	-5.1
Total	**4.0**	**10.2**	**9.8**	**12.7**	**8.4**	**8.5**

- Still noisy: with 512k paths $\approx$ same accuracy as smoothed version with 32k paths

⇨ Need new Theta + smoothing

Splitting of Theta

- We know how to split Theta into Vanilla-like and exotic-like Thetas – cf 1st talk

$$\Theta \;=\; \Theta_{\text{vanilla}} + \Theta_{\text{exotic}}$$

  - $\Theta_{\text{vanilla}} \equiv$ Theta of (perfect) vanilla hedge portfolio
  - Θ_{exotic} vanishes if derivative $\equiv$ static portfolio of vanilla options

- Recipe for Θ computation with time-t prices
  - Insert artificial δt maturity in market smile
  - Calculate 3 time-t prices with vol surfaces $\widehat{\sigma}_{KT}, \quad \widehat{\sigma}^{\dagger}_{KT}, \quad \widehat{\sigma}^{\star}_{KT}$

$$\Theta_{\text{exotic}} \;=\; \frac{1}{\delta t}\Big[P(t,S,\widehat{\sigma}^{\dagger}_{KT}) - P(t,S,\widehat{\sigma}_{KT})\Big]$$

$$\Theta_{\text{vanilla}} \;=\; \frac{1}{\delta t}\Big[P(t,S,\widehat{\sigma}^{\star}_{KT}) - P(t,S,\widehat{\sigma}^{\dagger}_{KT})\Big]$$

- Autocall example: $\delta t = 7$ days, $\epsilon = 0.5$, 512k MC paths

	Coupon 1	Coupon 2	Coupon 3	Coupon 4	Coupon 5	Coupon 6	100 + PDI	Total
Vanilla	25.2	-0.3	-3.5	-1.0	-5.0	-3.5	-4.0	**7.8**
Exotic	0.06	-0.87	0.68	-0.43	0.40	-0.16	-0.09	**-0.4**
Total	25.3	-1.2	-2.8	-1.5	-4.6	-3.7	-4.1	**7.4**

Gaps of barriers

- What else can we do with smooth(ed) barriers?
- Calculate effective step sizes of recall clauses: difference between continuation and exercise/recall values
- Autocall live at $T_i \equiv$ a European payoff of maturity T_i that pays:
  - continuation value if $S_i < L_i$
  - recall value if $S_i > L_i$
- Move barrier L_i slightly up
  - price moves up ⇨ Continuation value $>$ recall value
  - price moves down ⇨ Continuation value $<$ recall value
- Can access size of the discontinuity + how it depends on whether the KI barrier was hit or not

Step size of barrier option

- Consider autocall-type structure. Gets recalled at dates T_i if $S_i > L_i$

$S_i < L_i$ nothing happens – stay in product

$S_i > L_i$ client gets $c_i(S_i)$ – autocall terminates

- Indicator I_i that autocall is live at T_i:

$$I_i \;=\; \prod_{j<i}(1-\theta(S_j-L_j))$$

- Price of autocall can be written as:

$$P \;=\; P_i^- \;+\; E\Big[I_i\cdot\Big(\theta(S_i-L_i)c_i(S_i)+(1-\theta(S_i-L_i))f(S_i,U)\Big)\Big]$$

where

  - P_i^-: value of coupons paid before T_i
  - $f(S,U)$: continuation value. U discrete variable: 0 if DI triggered, 1 otherwise

- Take derivative wrt L_i

$$\begin{aligned}\frac{dP}{dL_i} &= E\Big[-\delta(S_i-L_i)\;I_i\;\big(c_i(S_i)-f_i(S_i,U)\big)\Big]\\ &= -E\Big[\delta(S_i-L_i)\;I_i\;\big(c_i(L_i)-f_i(L_i,U)\big)\Big]\end{aligned}$$

Step size – case 1: no path-dep variable: $P(t, S)$

- No U, f only a function of S

$$\begin{aligned}\frac{dP}{dL_i} &= -E\Big[\delta(S_i - L_i)\ I_i\ \big(c_i(L_i) - f_i(L_i)\big)\Big] \\ &= -(c_i - f_i)\ E\Big[\delta(S_i - L_i)\ I_i\Big]\end{aligned}$$

where $c_i - f_i \ \equiv\ c_i(L_i) - f_i(L_i)$

- D_i: price of option that pays 1 if (a) product still live at T_i, (b) $S_i > L_i$

$$\begin{aligned}D_i &= E\left[\theta\left(S_i - L_i\right) I_i\right] \\ \frac{dD_i}{dL_i} &= -\,E\left[\delta\left(S_i - L_i\right) I_i\right]\end{aligned}$$

- Recall step size $(c_i - f_i)$ is given by:

$$c_i - f_i \ = \ \frac{1}{\frac{dD_i}{dL_i}}\frac{dP}{dL_i}$$

⇨ Algo: for each L_i:

  - Price autocall P and digit D_i
  - Shift barrier L_i slightly and reprice both: $P + \Delta P$, $D_i + \Delta D_i$
  - Size of effective digital given by $c_i - f_i \ = \ \frac{\Delta P}{\Delta D_i}$

Step size – case 2: binary path-dep variable: $P(t, S, U)$

- $U_{t=0} = 1$. $U = 0$ as soon as DI barrier KI is triggered:

$$U_t \;=\; \prod_{k<t} \theta\left(S_k - KI\right)$$

- 2 continuation values $f_i(S, U) \;=\; U\, f_i^1(S) + (1 - U)\, f_i^0(S)$
- 2 effective step sizes depending upon whether KI barrier was hit:

$$\begin{aligned} c_i - f_i^0 &= c(L_i) - f_i^0(L_i) \\ c_i - f_i^1 &= c(L_i) - f_i^1(L_i) \end{aligned}$$

- Measure sensitivities to barrier shift of:
  - Regular untriggered autocall: $P^1(S) \;=\; P(S, U = 1)$
  - KI-triggered autocall: $P^0(S) \;=\; P(S, U = 0)$
- Options of maturity T_i that pay 1 if (a) product is still live, (b) $S_i > L_i$, (c) DI barrier has been triggered or not:

$$\begin{aligned} D_i^1 &= E\left[\theta(S_i - L_i)\; I_i U_i\right] \\ D_i^0 &= E\left[\theta(S_i - L_i)\; I_i (1 - U_i)\right] \end{aligned}$$

Step size – case 2: binary path-dep variable: $P(t, S, U)$ – 2

- Sensitivities of D_i^0, D_i^1 to barrier shift:

$$\frac{dD_i^1}{dL_i} = E\left[-\delta\left(S_i - L_i\right)\ I_i U_i\right]$$
$$\frac{dD_i^0}{dL_i} = E\left[-\delta\left(S_i - L_i\right)\ I_i\left(1 - U_i\right)\right]$$

- Sensitivities of untriggered / KI-triggered autocall:

$$\begin{aligned}
\frac{dP^1}{dL_i} &= E\Big[-\delta\left(S_i - L_i\right)\ I_i\ \left(c_i\left(S_i\right) - f_i\left(S_i, U_i\right)\right)\Big] \\
&= E\Big[-\delta(S_i - L_i)\ I_i\ \Big(c_i\left(S_i\right) - \big(\left(1 - U_i\right) f_i^0\left(S_i\right) + U_i f_i^1\left(S_i\right)\big)\Big)\Big] \\
&= (c_i - f_i^0)E\Big[-\delta\left(S_i - L_i\right) I_i\left(1 - U_i\right)\Big] + (c_i - f_i^1)E\Big[-\delta\left(S_i - L_i\right) I_i U_i\Big] \\
&= (c_i - f_i^0)\frac{dD_i^0}{dL_i} + (c_i - f_i^1)\frac{dD_i^1}{dL_i}
\end{aligned}$$

$$\begin{aligned}
\frac{dP^0}{dL_i} &= \left(c_i - f_i^0\right) E\Big[-\delta\left(S_i - L_i\right) I_i\Big] \\
&= \left(c_i - f_i^0\right) E\Big[-\delta\left(S_i - L_i\right) I_i\left(1 - U_i + U_i\right)\Big] \\
&= \left(c_i - f_i^0\right)\left(\frac{dD_i^0}{dL_i} + \frac{dD_i^1}{dL_i}\right)
\end{aligned}$$

Step size – case 2: binary path-dep variable: $P(t, S, U)$ – 3

- Have everything we need:

$$\begin{aligned}\frac{dP^1}{dL_i} &= \left(c_i - f_i^0\right)\frac{dD_i^0}{dL_i} + \left(c_i - f_i^1\right)\frac{dD_i^1}{dL_i} \\ \frac{dP^0}{dL_i} &= \left(c_i - f_i^0\right)\left(\frac{dD_i^0}{dL_i} + \frac{dD_i^1}{dL_i}\right)\end{aligned}$$

⇨ Get our two step sizes

$$\begin{aligned}c_i - f_i^0 &= \frac{\Delta P^0}{\Delta D_i^0 + \Delta D_i^1} \\ c_i - f_i^1 &= \frac{1}{\Delta D_i^0 + \Delta D_i^1}\left(\Delta P^1 + \frac{\Delta D_i^0}{\Delta D_i^1}\left(\Delta P^1 - \Delta P^0\right)\right)\end{aligned}$$

- What if we compute just one size $c_i - f_i^\star$ as in case 1, irrespective of triggering of KI barrier?

$$c_i - f_i^\star = \frac{\Delta P^1}{\Delta D_i^0 + \Delta D_i^1} = (c_i - f_i^0)\frac{\Delta D^0}{\Delta D_i^0 + \Delta D_i^1} + (c_i - f_i^1)\frac{\Delta D^1}{\Delta D_i^0 + \Delta D_i^1}$$

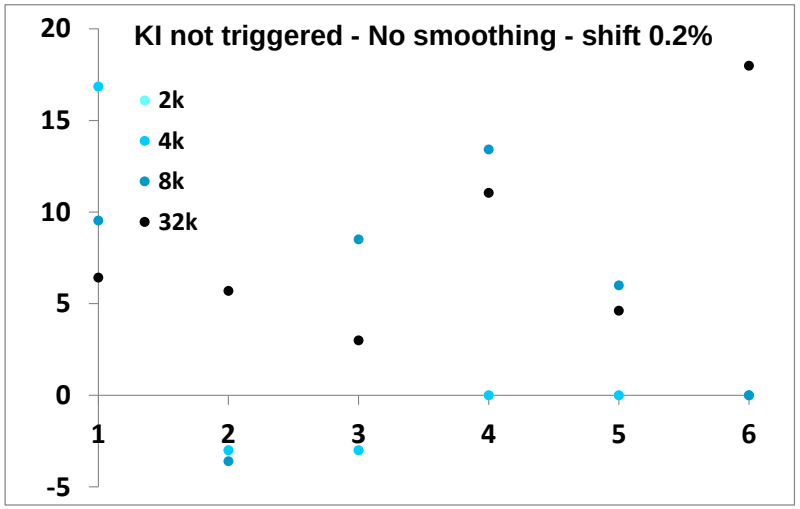
- If spots $\gg$ KI barrier, $D^0 \approx 0$: $c_i - f_i^\star \approx c_i - f_i^1$

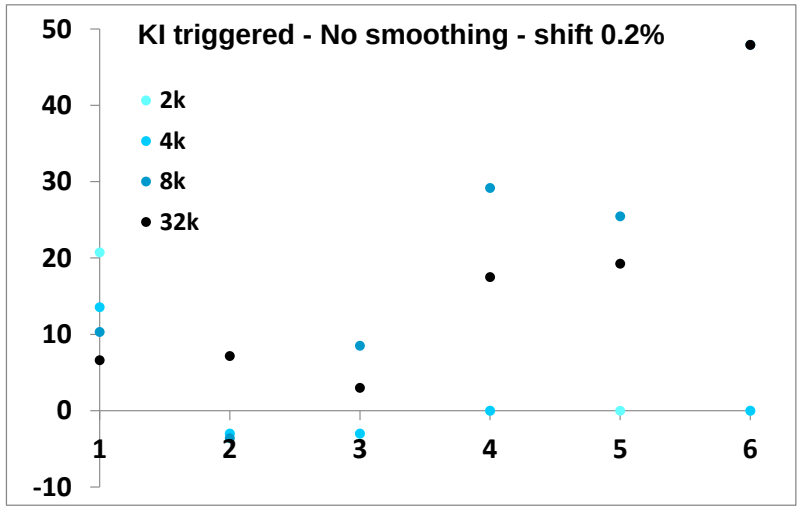
Autocalls: effective step sizes at recalls

- 3y autocall on Worst-of of Eurostoxx50/SP500/NIKKEI
- Recall every 6 months. Recall barriers L_i: 90, 90, 85, 80, 75, 70
- If recalled at T_i get $(1 + c_i)$ and autocall terminates. Coupon levels c_i: 3%, 6%, 9%, 12%, 15%, 18%
- At maturity get coupon & principal minus DI ATM put with DI barrier $L_{DI} = 60\%$
- DI barrier sampled at dates t_l for simplicity
- Market smiles as of 21 Feb 2017 – no rates/no repos
- We know the barrier sizes at maturity – recall barrier = 70
  - If DI barrier not triggered, discontinuity $= 118 - 100 = 18$
  - If DI barrier triggered, discontinuity $= 118 - 70 = 48$

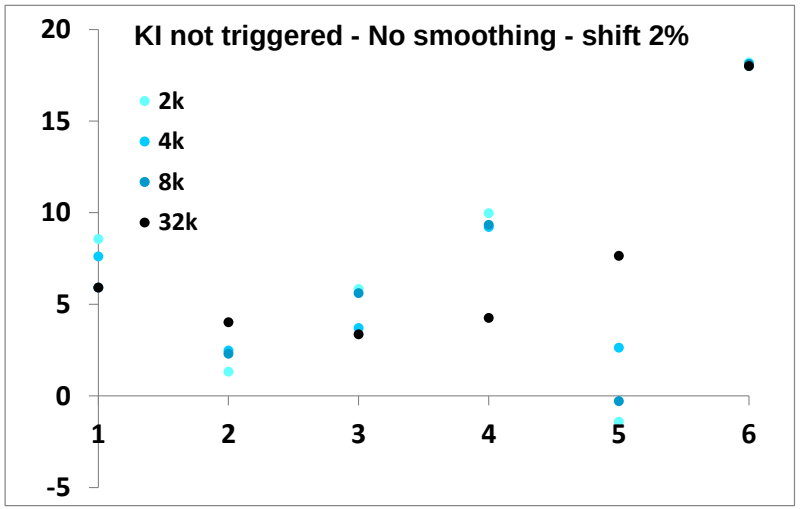
Autocalls – tests – 1

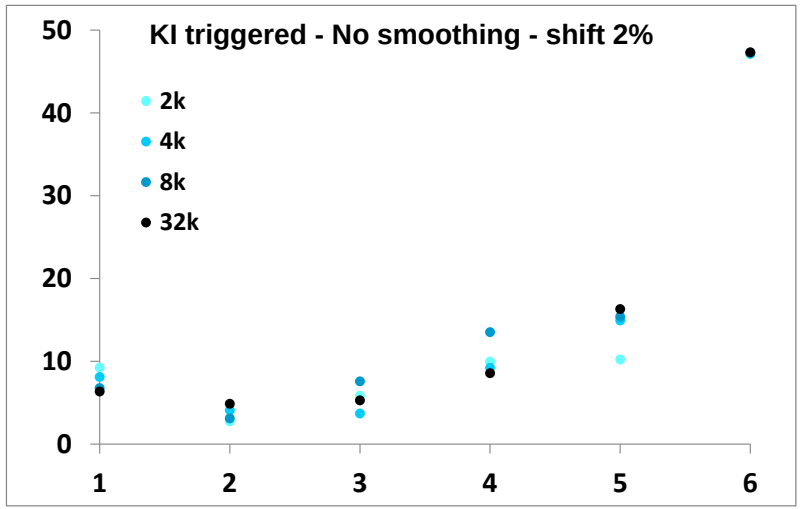
- First, no smoothing
- Results with 2k, 4k, 8k, 32k MC paths
- Shift recall barriers by 0.2%





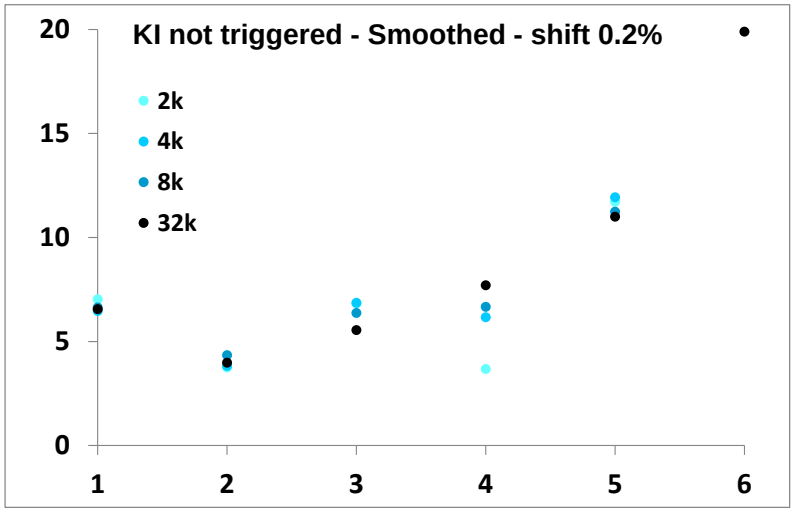
- Shift recall barriers by 2%

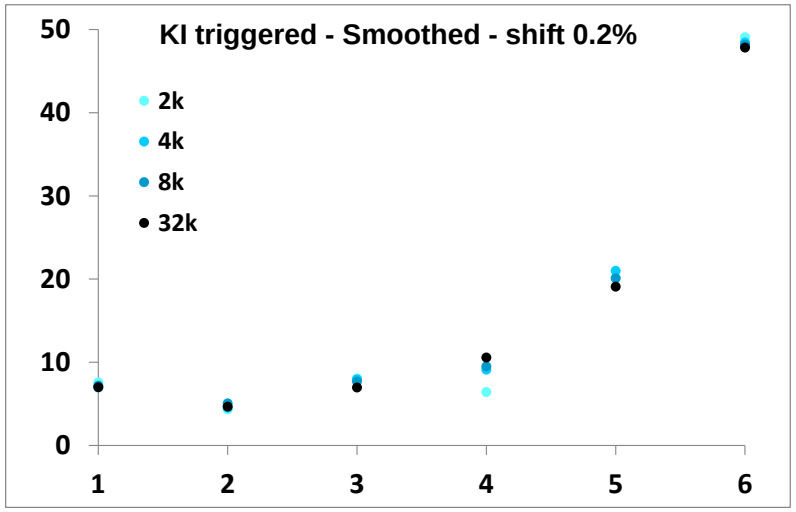




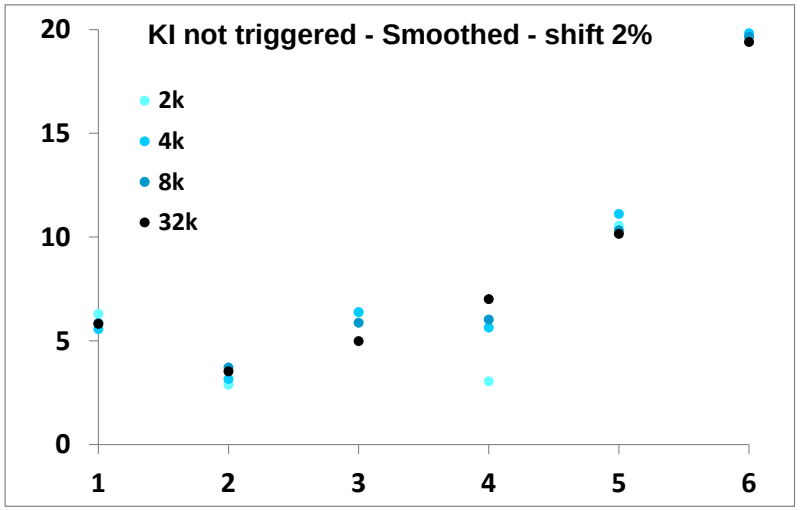
Autocalls – tests – 2

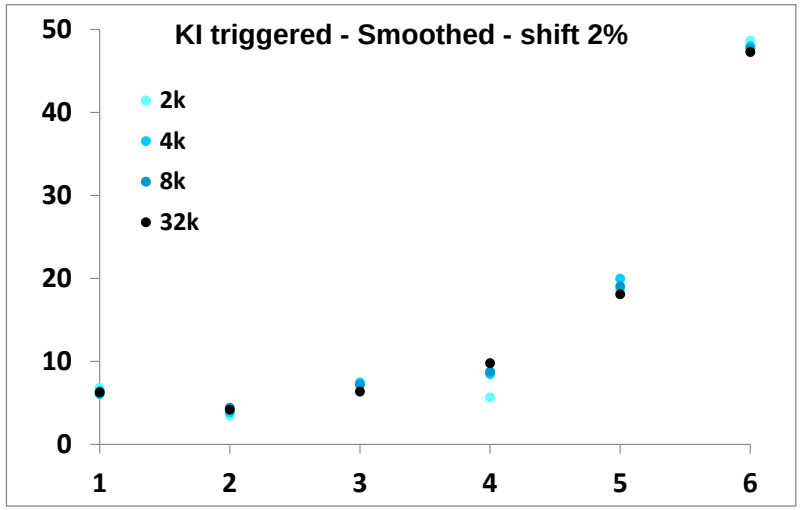
- Now with smoothing
- Results with 2k, 4k, 8k, 32k MC paths
- Shift recall barriers by 0.2%





- Shift recall barriers by 2%





⇨ Cleaner – 0.2% shift is OK

Smoothing arbitary payoffs – 1

- We know how to smooth out digitals
- What about other payoffs ?
  - In practice replace digitals with narrow Call Spreads. How to smooth out CSs?
- Any payoff $f(S)$can be written as a sum of digitals

$$\begin{aligned} f(S) &= f(0) + \int_0^\infty \left(\frac{df}{dL}\right) \theta(S-L) dL \\ &= f(0) + \int_0^\infty \left(\frac{df}{dL}\right) \theta\left(1 - \frac{L}{S}\right) dL \end{aligned}$$

- Replace Heaviside function with smoothed version:

$$\theta(x) \quad \rightarrow \quad \int \varphi_\delta(u)\ \theta\left(x-u\right) du$$

- Get smoothed payoff:

$$\begin{aligned} f_\delta(S) &= \int \varphi_\delta(u)\ f\big(S(1-u)\big) du \\ &= \int h(z)\ f\big(S(1-\delta z)\big) dz \end{aligned}$$

Smoothing arbitary payoffs – 2

- Effect on affine payoff $f(S) = aS + b$? $\int h(z)dz = 1$, $\int h(z)zdz = 0$. Thus:

$$f_\delta(S) = f(S)$$

- Also holds for piecewise affine $f(S)$, away from kinks, if h has finite support
  - What if we had used $\theta\left(\frac{S}{L} - 1\right)$ instead of $\theta(1 - \frac{L}{S})$?

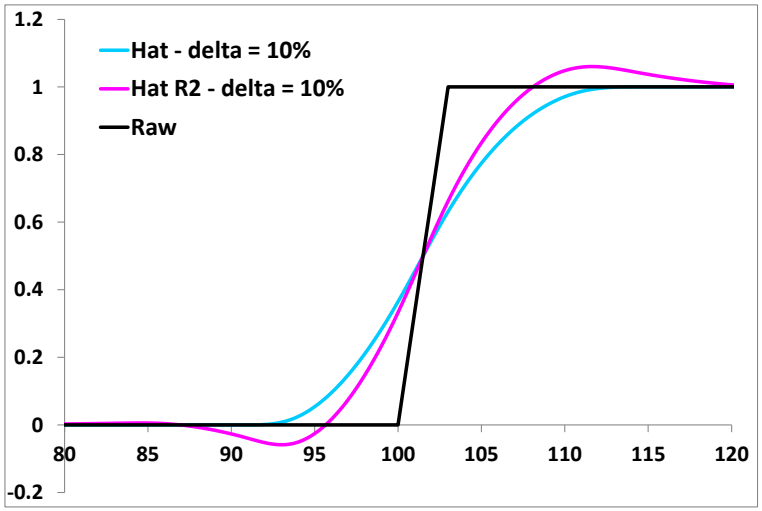
$$f_\delta(S) = \int h(z)\, f\left(\frac{S}{1+\delta z}\right) dz \neq f(S)$$

- $f_\delta(S)$ for Call payoff with hat kernel?

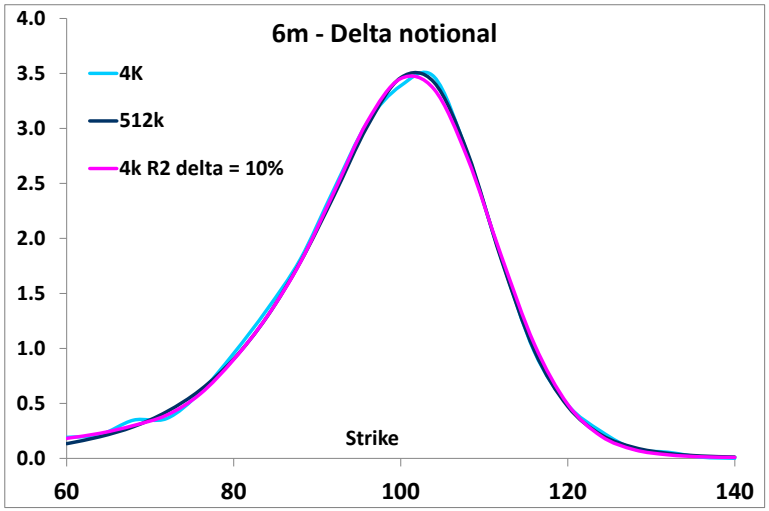
$$\begin{array}{ll}
[0, \dfrac{K}{1+\delta}] & 0 \\
[\dfrac{K}{1+\delta}, K] & \dfrac{1}{6}\dfrac{((1+\delta)S - K)^3}{(\delta S)^2} \\
[K, \dfrac{K}{1-\delta}] & (S-K) + \dfrac{1}{6}\dfrac{(K - S(1-\delta))^3}{(\delta S)^2} \\
[\dfrac{K}{1-\delta}, +\infty] & (S-K)
\end{array}$$

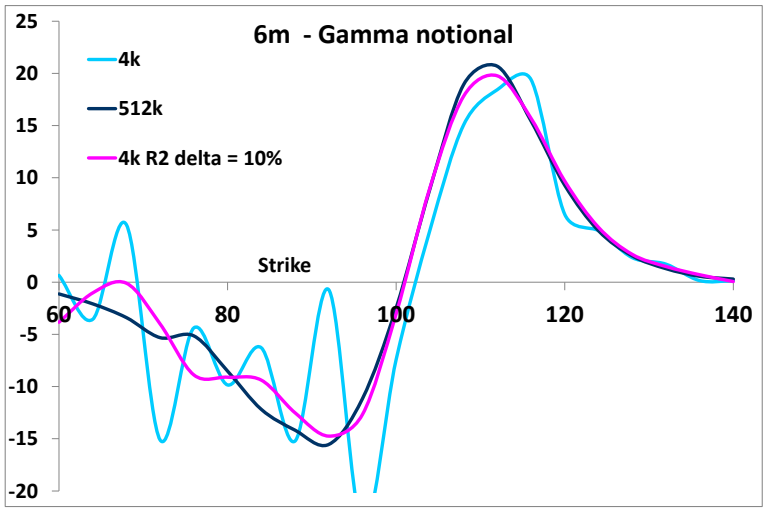
Smoothing arbitary payoffs – 3

- Look at binary of strike L, replaced by $[L,\ L(1+\epsilon)]$ Call spread
- Compare smoothed/unsmoothed CS. Here $\epsilon = 3\%$



- Delta/Gamma of 6m smoothed/unsmoothed Call spreads , K from 60% to 140%:





⇨ Even a 6m 3%-wide Call spread needs smoothing

Conclusion

- Smoothing of barriers yields better greeks: Thetas & Gammas
- Easy: replace raw Heaviside step function with R2-smoothed version
  - Smoothing parameter depends on implied vols/time to maturity
  - ⇨ Optimal smoothing policy needs tweaking
  - ⇨ Especially when multiplying multiple correlated Heaviside functions:

$$\theta^2(x) \;=\; \theta(x) \;\text{ but }\; \theta_\delta^2(x) \;\neq\; \theta_\delta(x)$$

- Cheap & easy estimation of effective step sizes
  - No need for regression
  - Only need continuation value for one spot value: $S =$ recall barrier, not full profile
- Determine effective step sizes working backwards in recall schedule
- Once KI and no-KI barrier sizes are determined, shift/adjust barrier and move down to previous recall
- Once effective digitals are replaced with call spreads, smooth out latter as well