Theta (and other Greeks): smooth barriers

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SOCIETE GENERALE

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Outline

- Test of new Theta technique on digitals
- Smoothing digitals
- Deltas & Gammas of digitals
- Greeks of autocalls
- Determining gaps of recall barriers of autocalls
Try new Theta technique (cf Talk 1) on European digitals

  - Standard technique $P(t), P(t + 1)$ day with 32k, 512k paths
  - New method $\delta t = 5$ tdays.

- Annualized Theta of digitals – strikes [60, 140], mat = 20 Oct 17 – 6 months

- Annualized Theta of digitals – strikes [20, 220], mat = 17 Dec 21 – 4 years

ɦ New method not good enough – greeks of digitals too noisy Ɇ do something
Smoothing barriers – 1

- Start with digital option that pays $f(S)$ if $S > L$, 0 otherwise

\[ f(S)\theta(S - L) = f(S)\theta(x) \quad \text{with} \quad x = 1 - \frac{L}{S} \]
\[ \theta(x) = 1_{x \geq 0} \]

- Replace $\theta(x)$ with smooth version $\theta_\delta(x)$, smoothed over typical width $\delta$, using kernel $\varphi_\delta$:

\[ \theta(x) \rightarrow \theta_\delta(x) = \int \varphi_\delta(u) \theta(x - u) \, du \]

- $\varphi_\delta$ given by (a) shape function $h$, (b) smoothing parameter $\delta$:

\[ \varphi_\delta(u) = \frac{1}{\delta} h \left( \frac{u}{\delta} \right) \]

- Smoothed barrier $\theta_\delta(x)$:

\[ \theta_\delta(x) = H \left( \frac{x}{\delta} \right) \quad \text{with} \quad H(w) = \int_{-\infty}^{w} h(z) \, dz \]
Smoothing barriers – 2

- $P_\delta(L)$: price of barrier option with smoothed barrier $\theta_\delta$

\[
P_\delta(L) = E\left[f(S) \theta_\delta\left(1 - \frac{L}{S}\right)\right] = E\left[f(S) \int \varphi_\delta(u) \theta\left(1 - \frac{L}{S} - u\right) du\right]
\]

\[
= \int \varphi_\delta(u) E\left[f(S) \theta\left(1 - \frac{L}{S(1-u)}\right)\right] du = \int \varphi_\delta(u) P\left(\frac{L}{1-u}\right) du
\]

\[
= \int h(z) P\left(\frac{L}{1-\delta z}\right) dz
\]

where $P(L)$ is price of standard barrier option: $P(L) = E\left[f(S) \theta\left(1 - \frac{L}{S}\right)\right]$

- Expand in $\delta$:

\[
P_\delta(L) = \int h(z) \left( (P(L) + a_1(L)\delta z + \frac{1}{2}a_2(L)\delta^2 z^2 + ... ) \right) dz
\]

\[
= P(L) \int h(z) dz + a_1(L) \overline{z} \cdot \delta + \frac{1}{2}a_2(L) \overline{z^2} \cdot \delta^2 + ...
\]

with $\overline{z^k} = \int h(z) z^k dz$

\[
a_1(L) = LP', \ a_2(L) = 2LP' + L^2 P'', \ a_3(L) = 6LP' + 6L^2 P''' + L^3 P''', \ a_4(L) = 24LP' + 36L^2 P'' + 12L^3 P''' + L^4 P'''''
\]
Smoothing barriers – 3

- No order-zero bias \( \Rightarrow \int h(z) dz = 1 \) (\( h \) density – if positive)

- No order-one bias \( \Rightarrow \bar{z} = \int z h(z) dz = 0 \)

- Error starts at order 2: \( \delta^2 \bar{z}^2 = \delta^2 \int h(z) z^2 dz > 0 \)

- Remove 2nd order error by using 2 different smoothings:

\[
P_{\delta R}^R = \frac{1}{3} (4P_{\delta} - P_{2\delta})
\]

  - No need to run 2 MCs. Run 1 MC with \( \theta(x) \) replaced with \( \theta_{\delta R}(x) \) given by:

\[
\theta_{\delta R} = \frac{1}{3} (4\theta_{\delta} - \theta_{2\delta})
\]

  - If \( h \) symmetric get order-3 accuracy for free, otherwise use:

\[
\theta_{\delta R2} = \frac{1}{21} (32\theta_{\delta} - 12\theta_{2\delta} + \theta_{4\delta})
\]

  - If \( h \) symmetric:

\[
P_{\delta R}^R = P - \frac{1}{6} a_4 \bar{z}^4 \delta^4 + ... \\
P_{\delta R2}^R = P + \frac{4}{21} a_4 \bar{z}^4 \delta^4 + ...
\]
Smoothing barriers – 4

▶ Check out different kernels (shape functions)

\[ \theta(x) \rightarrow \theta_\delta(x) = \int \varphi_\delta(u)\theta(x-u)du = H\left(\frac{x}{\delta}\right) \]

▶ Hat

<table>
<thead>
<tr>
<th>( z )</th>
<th>( ] -\infty, -1[ )</th>
<th>( [-1, 0] )</th>
<th>( [0, 1] )</th>
<th>( [1, \infty[ )</th>
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<tbody>
<tr>
<td>( h )</td>
<td>0</td>
<td>( 1+z )</td>
<td>( 1-z )</td>
<td>0</td>
</tr>
<tr>
<td>( H )</td>
<td>0</td>
<td>( \frac{1}{2}(1+z)^2 )</td>
<td>( 1-\frac{1}{2}(1-z)^2 )</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ \overline{z^2} = \frac{1}{6} \quad \overline{z^4} = \frac{1}{15} \]

▶ Gaussian

\[ h(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad H(z) = \mathcal{N}(z) \]

\[ \overline{z^2} = 1 \quad \overline{z^4} = 3 \]

⇒ \( \text{Hat}(\delta) \approx \text{Gaussian}\left(\frac{\delta}{\sqrt{6}}\right) \)
Smoothing barriers – 5

- Logit

\[ h(z) = \frac{e^{-z}}{(e^{-z} + 1)^2} \]
\[ H(z) = \frac{1}{1 + e^{-z}} \]
\[ \bar{z}^2 = \frac{\pi^2}{3} \]
\[ \bar{z}^4 = \frac{7\pi^4}{15} \]

\[ \hat{\delta} \approx \logit\left( \frac{\delta}{\pi \sqrt{2}} \right) \]

- In practice kernels are similar. Compare \( \hat{\delta} \), \( \text{Gaussian}(\frac{\delta}{\sqrt{6}}) \), \( \logit(\frac{\delta}{\pi \sqrt{2}}) \):

- Same kernels applied to \( x = \ln \frac{S}{L} \) rather than \( x = 1 - \frac{L}{S} \) \( \Rightarrow \) similar as well
Smoothing barriers – 6

▶ What if multiple barriers (Autocall)? Payoff can always be written as:

\[ f(S) \prod_i \theta(S_i - L_i) \]

⇒ Use smoothed version of digits:

\[ f(S) \prod_i \theta \left( S_i - L_i \right) = f(S) \cdot \prod_i \varphi \left( u_i \right) \cdot \theta \left( 1 - \frac{L_i}{S_i} - u_i \right) \, du_i \]

▶ Take expectation – smoothed price given by:

\[ P_\delta(L) = \int \prod_i \varphi \left( u_i \right) \, du_i \cdot P \left( \frac{L}{1-u} \right) \]

▶ Expand in powers of \( u_i \). At 2nd order get usual diagonal terms + cross-terms:

\[ L_i L_j \cdot \frac{d^2 P}{dL_idL_j} \int \varphi \left( u_i \right) u_i \, du_i \int \varphi \left( u_j \right) u_j \, du_j = L_i L_j \cdot \frac{d^2 P}{dL_idL_j} \cdot \frac{\bar{u_i} \, \bar{u_j}}{\bar{u} \, \bar{u}} = 0, \text{if } h \text{ symmetric} \]

▶ Probabilistic interpretation of smoothing algo

▶ Barrier \( L_i \)  ⇒ Random barrier \( \frac{L_i}{1 - \delta_i X_i} \)

▶ Random variables \( X_i \) independent and distributed with density \( h \)

▶ To achieve 2nd order accuracy, run 2 MCs with \( \delta, \, 2\delta \).
Smoothing barriers – bias – 1

Examples with $\delta = 20\%$: Hat (left) and Gaussian (right) kernels very similar

Black-Scholes, $\sigma = 20\%$, LogHat kernel with $\delta = 10\%$
$S = 100$, $T = 6$ months. Difference ($P - P_\delta$) for digitals struck over [60%, 140%]
Smoothing digitals – bias – 2

- **$T = 6m$ (SX5E, 21 Feb 17):** errors wrt ”exact prices” (512k paths). $\sigma_{ATM} = 17\%$

- **$T = 1m$:** $\sigma_{ATM} = 12\%$

⇒ R2 somewhat better than R. How to pick $\delta$ ?
Smoothing digitals – bias – 3

- Smoothing (or randomizing) barrier ⇒ additional contribution to volatility
- In $6m/\delta = 10\%$ and $1m/\delta = 2.5\%$ examples, additional variance: $\frac{\delta^2_T}{6\sigma^2_T T} \approx 10\%$

- Different kernels yield similar biases ⇒ in what follows use hat kernel
Smoothing digitals – $\Delta$, $\Gamma$

$\triangleright$ 4k $\equiv$ 4k MC paths

$\rightarrow$ $\delta = 5\%$: too large

$\triangleright$ Particularly efficient for longer maturities
Smoothing barriers – $\Theta – 1$

- Compare New, New with smoothing – 32k paths.
- Take $\delta t = 10$ days (1st maturity), $\epsilon = 50\%$. Hat kernel. $\delta_T = \sqrt{10\% \cdot 6\hat{\sigma}^2_T}$

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- OK, works. Now move on to autocalls
Autocalls – 1

- 3y autocall on Worst-of of Eurostoxx50/SP500/NIKKEI
- Recall every 6 months. Recall barriers $L_i$: 90, 90, 85, 80, 75, 60
- If recalled at $T_i$ get $(1 + c_i)$ and autocall terminates. Coupon levels $c_i$: 3%, 6%, 9%, 12%, 15%, 18%
- At maturity get coupon & principal minus DI ATM put with DI barrier $L_{DI} = 60$
- DI barrier sampled at dates $t_i$ for simplicity
- Market smiles as of 21 Feb 2017 – no rates/no repos

Pricing with smoothed barriers

- No more if then else clauses in scripts
- Rewrite payoffs using $\theta$ functions, then replace with smoothed version $\phi$
- Introduce (a) survival ”probability” $p_i$ at $T_i$, (b) DI trigger ”probability” $p_{DI}$
- One underlying for simplicity:

\[
p_i = \prod_{j<i} \left(1 - \theta \delta j \left(1 - \frac{L_j}{S_j}\right)\right)
\]

\[
p_{DI} = p_n \left(1 - \prod I \theta I \left(1 - \frac{L_{DI}}{S_I}\right)\right)
\]

- Payoff at recall dates $T_i$, including maturity $T_n$: $(1 + c_i)p_i$  
  + extra payoff at $T_n$ if $S_n < L_n$: $p_{n+1} \cdot 1 - p_{DI} (1 - S_n)^+$
Autocalls – 2

- Autocall is written on Worst-Of
  - Solution 1: evaluate smoothed barrier on Worst-of:
    \[
    \theta \left(1 - \frac{L}{\min_k S^k}\right) \Rightarrow \theta_\delta \left(1 - \frac{L}{\min_k S^k}\right)
    \]
    - Works but does not allow for individual setting of smoothing parameter \(\delta\)
  - Solution 2: smooth barriers individually:
    \[
    \theta \left(1 - \frac{L}{\min_k S^k}\right) = \prod_k \theta \left(1 - \frac{L}{S^k}\right) \Rightarrow \prod_k \theta_\delta \left(1 - \frac{L}{S^k}\right)
    \]
    - Individually adjust \(\delta\) to each underlying’s vol level

- In numerical tests use Solution 1, hat kernel, \(\delta_T = \sqrt{10\% \cdot 6\hat{\sigma}_T^2 T}. \hat{\sigma}_T \approx 18\%\).

- Look at individual coupons + + split final payout into coupon + 100 - DI Put.
Autocalls – tests – Delta/Gamma

► Individual coupon prices: std vs smoothed barriers

<table>
<thead>
<tr>
<th>No smoothing</th>
<th>Coupon 1</th>
<th>Coupon 2</th>
<th>Coupon 3</th>
<th>Coupon 4</th>
<th>Coupon 5</th>
<th>Coupon 6</th>
<th>100 · PDI</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>8k</td>
<td>71.30</td>
<td>8.46</td>
<td>5.07</td>
<td>3.39</td>
<td>2.50</td>
<td>4.23</td>
<td>3.29</td>
<td>98.25</td>
</tr>
<tr>
<td>512k</td>
<td>71.13</td>
<td>8.39</td>
<td>5.40</td>
<td>3.53</td>
<td>2.49</td>
<td>4.18</td>
<td>3.27</td>
<td>98.39</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>R2 10% eq. var</th>
<th>Coupon 1</th>
<th>Coupon 2</th>
<th>Coupon 3</th>
<th>Coupon 4</th>
<th>Coupon 5</th>
<th>Coupon 6</th>
<th>100 · PDI</th>
<th>Total</th>
</tr>
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<tbody>
<tr>
<td>8k</td>
<td>71.44</td>
<td>8.19</td>
<td>5.21</td>
<td>3.53</td>
<td>2.55</td>
<td>4.04</td>
<td>3.31</td>
<td>98.28</td>
</tr>
<tr>
<td>512k</td>
<td>71.42</td>
<td>8.01</td>
<td>5.41</td>
<td>3.57</td>
<td>2.53</td>
<td>4.15</td>
<td>3.32</td>
<td>98.41</td>
</tr>
</tbody>
</table>

► Delta notionals $S_i \frac{dP}{dS_i}$ – 8k paths:

► Diagonal gamma notionals $S_i^2 \frac{d^2P}{dS_i^2}$ – 8k paths:
Autocalls – tests – Theta

- Annualized theta
  - Std: price at $t$, then at $t+2$ days: 8k, 32k, 512k, 2M paths
  - New: both prices at $t$ + smoothing. $\delta t = 7$ days, $\epsilon = 50\%$, no artificial maturity inserted

$$\Theta = \frac{P(\hat{\sigma}_k^2) - P(\hat{\sigma}_k^2 - \epsilon \frac{\delta t}{T_k} \hat{\sigma}_k^2)}{\epsilon \delta t}$$

<table>
<thead>
<tr>
<th>Standard</th>
<th>New - smoothed</th>
</tr>
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<tbody>
<tr>
<td>8k</td>
<td>32k</td>
</tr>
<tr>
<td>Coupon 1</td>
<td>39.0</td>
</tr>
<tr>
<td>Coupon 2</td>
<td>-23.6</td>
</tr>
<tr>
<td>Coupon 3</td>
<td>-12.1</td>
</tr>
<tr>
<td>Coupon 4</td>
<td>22.5</td>
</tr>
<tr>
<td>Coupon 5</td>
<td>2.6</td>
</tr>
<tr>
<td>Coupon 6</td>
<td>-34.2</td>
</tr>
<tr>
<td>100 - PDI</td>
<td>9.9</td>
</tr>
<tr>
<td>Total</td>
<td>4.0</td>
</tr>
</tbody>
</table>

- 1-day theta is indeed small quantity. $8/365 \approx 2$ bps/day.

⇒ New Theta + smoothing performs OK
Autocalls – tests – Theta – 2

- Is the efficiency coming from the new Theta technique or from barrier smoothing?

- Compute Theta with std technique, but with smoothed barriers

<table>
<thead>
<tr>
<th></th>
<th>Standard</th>
<th></th>
<th></th>
<th>Standard - smoothed</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8k</td>
<td>32k</td>
<td>512k</td>
<td>8k</td>
<td>32k</td>
<td>512k</td>
</tr>
<tr>
<td>Coupon 1</td>
<td>39.0</td>
<td>64.8</td>
<td>39.0</td>
<td>45.0</td>
<td>55.5</td>
<td>28.7</td>
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<tr>
<td>Coupon 2</td>
<td>-23.6</td>
<td>-60.8</td>
<td>-9.1</td>
<td>-14.1</td>
<td>-30.1</td>
<td>-2.4</td>
</tr>
<tr>
<td>Coupon 3</td>
<td>-12.1</td>
<td>10.3</td>
<td>-3.3</td>
<td>-4.2</td>
<td>-2.7</td>
<td>-1.7</td>
</tr>
<tr>
<td>Coupon 4</td>
<td>22.5</td>
<td>-3.7</td>
<td>-3.4</td>
<td>-11.3</td>
<td>-6.6</td>
<td>-2.1</td>
</tr>
<tr>
<td>Coupon 5</td>
<td>2.6</td>
<td>3.2</td>
<td>-2.0</td>
<td>4.8</td>
<td>0.2</td>
<td>-2.8</td>
</tr>
<tr>
<td>Coupon 6</td>
<td>-34.2</td>
<td>0.7</td>
<td>-5.6</td>
<td>-4.0</td>
<td>-3.8</td>
<td>-6.0</td>
</tr>
<tr>
<td>100 - PDI</td>
<td>9.9</td>
<td>-4.3</td>
<td>-5.7</td>
<td>-3.4</td>
<td>-4.1</td>
<td>-5.1</td>
</tr>
<tr>
<td>Total</td>
<td>4.0</td>
<td>10.2</td>
<td>9.8</td>
<td>12.7</td>
<td>8.4</td>
<td>8.5</td>
</tr>
</tbody>
</table>

- Still noisy: with 512k paths $\approx$ same accuracy as smoothed version with 32k paths

$\Rightarrow$ Need new Theta + smoothing
Splitting of Theta

- We know how to split Theta into Vanilla-like and exotic-like Thetas – cf 1st talk
  \[ \Theta = \Theta_{\text{vanilla}} + \Theta_{\text{exotic}} \]
  - \( \Theta_{\text{vanilla}} \equiv \) Theta of (perfect) vanilla hedge portfolio
  - \( \Theta_{\text{exotic}} \) vanishes if derivative \( \equiv \) static portfolio of vanilla options

- Recipe for \( \Theta \) computation with time-\( t \) prices
  - Insert artificial \( \delta t \) maturity in market smile
  - Calculate 3 time-\( t \) prices with vol surfaces \( \hat{\sigma}_{KT}, \hat{\sigma}^\dagger_{KT}, \hat{\sigma}^*_{KT} \)
  \[
  \Theta_{\text{exotic}} = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}^\dagger_{KT}) - P(t, S, \hat{\sigma}_{KT}) \right]
  \]
  \[
  \Theta_{\text{vanilla}} = \frac{1}{\delta t} \left[ P(t, S, \hat{\sigma}^*_{KT}) - P(t, S, \hat{\sigma}^\dagger_{KT}) \right]
  \]

- Autocall example: \( \delta t = 7 \) days, \( \epsilon = 0.5 \), 512k MC paths

<table>
<thead>
<tr>
<th></th>
<th>Coupon 1</th>
<th>Coupon 2</th>
<th>Coupon 3</th>
<th>Coupon 4</th>
<th>Coupon 5</th>
<th>Coupon 6</th>
<th>100 + PDI</th>
<th>Total</th>
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<tbody>
<tr>
<td>Vanilla</td>
<td>25.2</td>
<td>-0.3</td>
<td>-3.5</td>
<td>-1.0</td>
<td>-5.0</td>
<td>-3.5</td>
<td>-4.0</td>
<td>7.8</td>
</tr>
<tr>
<td>Exotic</td>
<td>0.06</td>
<td>-0.87</td>
<td>0.68</td>
<td>-0.43</td>
<td>0.40</td>
<td>-0.16</td>
<td>-0.09</td>
<td>-0.4</td>
</tr>
<tr>
<td>Total</td>
<td>25.3</td>
<td>-1.2</td>
<td>-2.8</td>
<td>-1.5</td>
<td>-4.6</td>
<td>-3.7</td>
<td>-4.1</td>
<td>7.4</td>
</tr>
</tbody>
</table>
Gaps of barriers

- What else can we do with smooth(ed) barriers?

- Calculate effective step sizes of recall clauses: difference between continuation and exercise/recall values

- Autocall live at $T_i \equiv a$ European payoff of maturity $T_i$ that pays:
  - continuation value if $S_i < L_i$
  - recall value if $S_i > L_i$

- Move barrier $L_i$ slightly up
  - price moves up $\Rightarrow$ Continuation value $> \text{ recall value}$
  - price moves down $\Rightarrow$ Continuation value $< \text{ recall value}$

- Can access size of the discontinuity $+$ how it depends on whether the KI barrier was hit or not
Step size of barrier option

- Consider autocall-type structure. Gets recalled at dates $T_i$ if $S_i > L_i$

  $S_i < L_i$ nothing happens – stay in product

  $S_i > L_i$ client gets $c_i(S_i)$ – autocall terminates

- Indicator $I_i$ that autocall is live at $T_i$:

  $$I_i = \prod_{j<i} (1 - \theta(S_j - L_j))$$

- Price of autocall can be written as:

  $$P_i = P_i^- + E \left[ I_i \cdot \left( \theta(S_i - L_i)c_i(S_i) + (1 - \theta(S_i - L_i))f(S_i, U) \right) \right]$$

  where

  - $P_i^-$: value of coupons paid before $T_i$

  - $f(S, U)$: continuation value. $U$ discrete variable: 0 if DI triggered, 1 otherwise

- Take derivative wrt $L_i$

  $$\frac{dP}{dL_i} = E \left[ -\delta(S_i - L_i) I_i \left( c_i(S_i) - f_i(S_i, U) \right) \right]$$

  $$= -E \left[ \delta(S_i - L_i) I_i \left( c_i(L_i) - f_i(L_i, U) \right) \right]$$
Step size – case 1: no path-dep variable: \( P(t, S) \)

- No \( U, f \) only a function of \( S \)

\[
\frac{dP}{dL_i} = -E \left[ \delta(S_i - L_i) l_i \left( c_i(L_i) - f_i(L_i) \right) \right]
\]
\[
= -(c_i - f_i) \ E \left[ \delta(S_i - L_i) l_i \right]
\]

where \( c_i - f_i \equiv c_i(L_i) - f_i(L_i) \)

- \( D_i \): price of option that pays 1 if (a) product still live at \( T_i \), (b) \( S_i > L_i \)

\[
D_i = E \left[ \theta(S_i - L_i) l_i \right]
\]
\[
\frac{dD_i}{dL_i} = -E \left[ \delta(S_i - L_i) l_i \right]
\]

- Recall step size \((c_i - f_i)\) is given by:

\[
c_i - f_i = \frac{1}{dD_i/dL_i} \frac{dP}{dL_i}
\]

⇒ Algo: for each \( L_i \):

- Price autocall \( P \) and digit \( D_i \)
- Shift barrier \( L_i \) slightly and reprice both: \( P + \Delta P, D_i + \Delta D_i \)
- Size of effective digital given by \( c_i - f_i = \frac{\Delta P}{\Delta D_i} \)
Step size – case 2: binary path-dep variable: \( P(t, S, U) \)

- \( U_{t=0} = 1 \). \( U = 0 \) as soon as DI barrier \( KI \) is triggered:
  \[
  U_t = \prod_{k<t} \theta(S_k - KI)
  \]

- 2 continuation values \( f_i(S, U) = U f_i^1(S) + (1 - U) f_i^0(S) \)

- 2 effective step sizes depending upon whether \( KI \) barrier was hit:
  \[
  c_i - f_i^0 = c(L_i) - f_i^0(L_i)
  \]
  \[
  c_i - f_i^1 = c(L_i) - f_i^1(L_i)
  \]

- Measure sensitivities to barrier shift of:
  - Regular untriggered autocall: \( P^1(S) = P(S, U = 1) \)
  - KI-triggered autocall: \( P^0(S) = P(S, U = 0) \)

- Options of maturity \( T_i \) that pay 1 if (a) product is still live, (b) \( S_i > L_i \), (c) DI barrier has been triggered or not:
  \[
  D_i^1 = E[\theta(S_i - L_i) I_i U_i]
  \]
  \[
  D_i^0 = E[\theta(S_i - L_i) I_i (1 - U_i)]
  \]
Step size – case 2: binary path-dep variable: \( P(t, S, U) - 2 \)

- Sensitivities of \( D_i^0, D_i^1 \) to barrier shift:

\[
\frac{dD_i^1}{dL_i} = E \left[ -\delta (S_i - L_i) \right] I_i U_i
\]
\[
\frac{dD_i^0}{dL_i} = E \left[ -\delta (S_i - L_i) \right] I_i (1 - U_i)
\]

- Sensitivities of untriggered / KI-triggered autocall:

\[
\frac{dP_i^1}{dL_i} = E \left[ -\delta (S_i - L_i) \right] I_i \left( c_i (S_i) - f_i (S_i, U_i) \right)
\]
\[
= E \left[ -\delta(S_i - L_i) \right] I_i \left( c_i (S_i) - \left( (1 - U_i) f_i^0 (S_i) + U_i f_i^1 (S_i) \right) \right)
\]
\[
= (c_i - f_i^0)E \left[ -\delta (S_i - L_i) \right] I_i (1 - U_i) + (c_i - f_i^1)E \left[ -\delta (S_i - L_i) \right] I_i U_i
\]
\[
= (c_i - f_i^0) \frac{dD_i^0}{dL_i} + (c_i - f_i^1) \frac{dD_i^1}{dL_i}
\]

\[
\frac{dP_i^0}{dL_i} = (c_i - f_i^0) E \left[ -\delta (S_i - L_i) \right] I_i
\]
\[
= (c_i - f_i^0) E \left[ -\delta (S_i - L_i) \right] I_i (1 - U_i) + U_i)
\]
\[
= (c_i - f_i^0) \left( \frac{dD_i^0}{dL_i} + \frac{dD_i^1}{dL_i} \right)
\]
Step size – case 2: binary path-dep variable: $P(t, S, U) – 3$

- Have everything we need:

$$\frac{dP^1}{dL_i} = (c_i - f_i^0) \frac{dD^0_i}{dL_i} + (c_i - f_i^1) \frac{dD^1_i}{dL_i}$$

$$\frac{dP^0}{dL_i} = (c_i - f_i^0) \left( \frac{dD^0_i}{dL_i} + \frac{dD^1_i}{dL_i} \right)$$

- Get our two step sizes

$$c_i - f_i^0 = \frac{\Delta P^0}{\Delta D^0_i + \Delta D^1_i}$$

$$c_i - f_i^1 = \frac{1}{\Delta D^0_i + \Delta D^1_i} \left( \Delta P^1 + \frac{\Delta D^0_i}{\Delta D^1_i} (\Delta P^1 - \Delta P^0) \right)$$

- What if we compute just one size $c_i - f_i^*$ as in case 1, irrespective of triggering of KI barrier?

$$c_i - f_i^* = \frac{\Delta P^1}{\Delta D^0_i + \Delta D^1_i} = (c_i - f_i^0) \frac{\Delta D^0_i}{\Delta D^0_i + \Delta D^1_i} + (c_i - f_i^1) \frac{\Delta D^1_i}{\Delta D^0_i + \Delta D^1_i}$$

- If spots $\gg$ KI barrier, $D^0 \approx 0$: $c_i - f_i^* \approx c_i - f_i^1$
Autocalls: effective step sizes at recalls

- 3y autocall on Worst-of of Eurostoxx50/SP500/NIKKEI
- Recall every 6 months. Recall barriers $L_i$: 90, 90, 85, 80, 75, 70
- If recalled at $T_i$ get $(1 + c_i)$ and autocall terminates. Coupon levels $c_i$: 3%, 6%, 9%, 12%, 15%, 18%
- At maturity get coupon & principal minus DI ATM put with DI barrier $L_{DI} = 60$
- DI barrier sampled at dates $t_i$ for simplicity
- Market smiles as of 21 Feb 2017 – no rates/no repos

- We know the barrier sizes at maturity – recall barrier = 70
  - If DI barrier not triggered, discontinuity = $118 - 100 = 18$
  - If DI barrier triggered, discontinuity = $118 - 70 = 48$
Autocalls – tests – 1

- First, no smoothing
- Results with 2k, 4k, 8k, 32k MC paths
- Shift recall barriers by 0.2%

- Shift recall barriers by 2%
Autocalls – tests – 2

▶ Now with smoothing
▶ Results with 2k, 4k, 8k, 32k MC paths
▶ Shift recall barriers by 0.2%

▶ Shift recall barriers by 2%

 Cleaner – 0.2% shift is OK
Smoothing arbitrary payoffs – 1

- We know how to smooth out digitals
- What about other payoffs?
  - In practice replace digitals with narrow Call Spreads. How to smooth out CSs?

- Any payoff $f(S)$ can be written as a sum of digitals

  $$ f(S) = f(0) + \int_0^\infty \left( \frac{df}{dL} \right) \theta(S - L) dL $$

  $$ = f(0) + \int_0^\infty \left( \frac{df}{dL} \right) \theta \left(1 - \frac{L}{S}\right) dL $$

- Replace Heaviside function with smoothed version:

  $$ \theta(x) \rightarrow \int \varphi_\delta(u) \theta(x - u) \, du $$

- Get smoothed payoff:

  $$ f_\delta(S) = \int \varphi_\delta(u) f(S(1 - u)) \, du $$

  $$ = \int h(z) f(S(1 - \delta z)) \, dz $$
Smoothing arbitrary payoffs – 2

- Effect on affine payoff \( f(S) = aS + b \)? \( \int h(z) dz = 1, \int h(z) zdz = 0. \) Thus:

\[
f_\delta(S) = f(S)
\]

- Also holds for piecewise affine \( f(S) \), away from kinks, if \( h \) has finite support

  - What if we had used \( \theta\left(\frac{S}{L} - 1\right) \) instead of \( \theta(1 - \frac{L}{S}) \)?

\[
f_\delta(S) = \int h(z) f\left(\frac{S}{1 + \delta z}\right) dz \neq f(S)
\]

- \( f_\delta(S) \) for Call payoff with hat kernel?

<table>
<thead>
<tr>
<th>Interval</th>
<th>Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, \frac{K}{1 + \delta}])</td>
<td>0</td>
</tr>
<tr>
<td>([\frac{K}{1 + \delta}, K])</td>
<td>(\frac{1}{6} \left((1 + \delta)S - K\right)^3)</td>
</tr>
<tr>
<td>([K, \frac{K}{1 - \delta}])</td>
<td>((S - K) + \frac{1}{6} \left(K - S(1 - \delta)\right)^3)</td>
</tr>
<tr>
<td>([\frac{K}{1 - \delta}, +\infty])</td>
<td>((S - K))</td>
</tr>
</tbody>
</table>
Smoothing arbitrary payoffs – 3

- Look at binary of strike $L$, replaced by $[L, L(1 + \epsilon)]$ Call spread
- Compare smoothed/unsmoothed CS. Here $\epsilon = 3\%$

Even a 6m 3%-wide Call spread needs smoothing
Conclusion

▶ Smoothing of barriers yields better greeks: Thetas & Gammas

▶ Easy: replace raw Heaviside step function with R2-smoothed version
  ▶ Smoothing parameter depends on implied vols/time to maturity
  ➔ Optimal smoothing policy needs tweaking
  ➔ Especially when multiplying multiple correlated Heaviside functions:
    \[ \theta^2(x) = \theta(x) \text{ but } \theta^2_\delta(x) \neq \theta_\delta(x) \]

▶ Cheap & easy estimation of effective step sizes
  ▶ No need for regression
    ▶ Only need continuation value for one spot value: \( S = \) recall barrier, not full profile

▶ Determine effective step sizes working backwards in recall schedule

▶ Once KI and no-KI barrier sizes are determined, shift/adjust barrier and move down to previous recall

▶ Once effective digitals are replaced with call spreads, smooth out latter as well